Mathematical analysis of generalized linear evolution equations with the non-singular kernel derivative

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Declaration of Authorship

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- Where any part of this dissertation has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this dissertation is entirely my own work.
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“Mathematical Analysis is as extensive as nature herself.”

Joseph Fourier

Abstract

Linear Evolution Equations (LEE) have been studied extensively over many years. Their extension in the field of fractional calculus have been defined by $D_t^\alpha u(x,t) = Au(x,t)$, where α is the fractional order and D_t^α is a generalized differential operator. Two types of generalized differential operators were applied to the LEE in the state-of-the-art, producing the Riemann-Liouville and the Caputo time fractional evolution equations. However the extension of the new Caputo-Fabrizio derivative (CFFD) to these equations has not been developed. This work investigates existing fractional derivative evolution equations and analyze the generalized linear evolution equations with non-singular kernel derivative. The well-posedness of the extended CFFD linear evolution equation is demonstrated by proving the existence of a solution, the uniqueness of the existing solution, and finally the continuous dependence of the behavior of the solution on the data and parameters. Extended evolution equations with CFFD are applied to kinetics, heat diffusion and dispersion of shallow water waves using MATLAB simulation software for validation purpose.

Key terms:

Mathematical analysis; Evolution equation; Caputo-Fabrizio derivative; Fractional derivarive; Diffusion equation; Fractional calculus; Non-singular kernel derivative; Solution operators; Semigroup; Well-posedness

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List of acronyms

ABFD	**A**tangana-**B**aleanu **F**ractional **D**erivative
BVP	**B**oundary **V**alue **P**roblem
CFD	**C**aputo **F**ractional **D**erivative
CFFD	**C**aputo-**F**abrizio **F**ractional **D**erivative
EE	**E**volution **E**quation
FC	**F**ractional **C**aculus
FDE	**F**ractional **D**ifferential **E**quation
IVP	**I**nitial **V**alue **P**roblem
LEE	**L**inear **E**volution **E**quation
LHAM	**L**aplace-**H**omotopy **A**nalysis **M**ethod
ML	**M**ittag-**L**effler function
ODE	**O**rdinary **D**ifferential **E**quation
PDE	**P**artial **D**ifferential **E**quation
RLFD	**R**iemann-**L**iouville **F**ractional **D**erivative

Dedicated to my younger brother Gauthier and my family; my wife Annie Claire and children Béatrice and Ivan. I love you all

Chapter 1

Introduction

"The art of doing mathematics consists in finding that special case which contains all the germs of generality."

David Hilbert

1.1 Introduction

Many natural phenomena in physical environment are most of the time governed by differential equations. A large number of processes uses systems which *evolve* with time to describe these phenomena in the field of applied sciences. An example of evolution equation is the homogeneous linear evolution equation defined by

$$\frac{du(x,t)}{dt} = Au(x,t) \tag{1.1}$$

where t, A, u are the time, a linear operator and a suitable function in a Banach space respectively. The function $u(x,t)$ is interpreted as a concentration at a point x and time t. Evolution Equations (EE) are collectively referred to as equations that describe the development of the concentration in time. Linear Evolution Equation (LEE) is a particular type of EE and it will be our study main focus. Here the ruling equation is linear. LEE have been studied extensively over many years. This study investigates on the existing LEE, their analysis in the field of Fractional calculus using fractional derivatives, and finally proposes an extension of New Caputo-Fabrizio derivative with fractional order, abbreviated CFFD for Caputo-Fabrizio Faractional Derivative, to LEE.

1.2 Problem statement

Fractional-order derivatives have been extensively studied for many years in various fields such as physics, engineering, applied mathematics including mathematical models and analysis of solutions of fractional differential equations characterizing the behaviors of dynamic systems. Considerable development in the field of fractional differential equations can be found in the literature [1–6]. The LEE is extended in the field of fractional calculus by the equation defined by

$$\frac{d^{\alpha}u(x,t)}{dt^{\alpha}} = Au(x,t) \tag{1.2}$$

where α is the fractional order and $\frac{d^{\alpha}u(x,t)}{dt^{\alpha}}$ also denoted by D_t^{α} is a generalized differential operator.

Two types of generalized differential operators were applied to the LEE in the state-of-the-art, producing the Riemann-Liouville and the Caputo time fractional evolution equations. Bazhlekova in [7, 8] has employed the old Caputo fractional derivative to model LEE by proving both existence and uniqueness of solutions. In [9], Bazhlekova has also analyzed the well-posedness of EE model based on the Riemann-Liouville time fractional derivative. Recently published work on well-posedness of completely monotone functions using both Riemann-Liouville and the Caputo time fractional derivatives was highlighted in [10]. Moreover, a particularity in the above-mentioned articles by Bazhlekova is that the existence of solution was established using the sub-ordination principle relating the solution operator of the considered problem.

However, the newly introduced CFFD has never been applied to Evolution Equations or LEE despite its numerous applications in physics, natural sciences and engineering [11–16]. An analysis of LEE modeled with CFFD will definitely enrich further the literature on Evolution Equations.

In this dissertation, we analyze Linear evolution equations modeled with the recently developed new Caputo-Fabrizio Fractional Derivative (CFFD), also referred to as the Fractional derivative with non-singular kernel. This implies that the problem of well-posedness related to the new CFFD model applied to LEE is addressed here.

1.3 Motivation

Originated from a complaint made on previous FDE whose mathematical expressions appeared cumbersome [14], the CFFD was developed as a simplified expression with no singular kernel. The use of no singular kernel removed difficulties previously experienced by the older fractional-order derivatives in solving fractional related models. Another benefit of the CFFD is its suitability for analysis tools such as Laplace and Fourier transforms [14], and also the effective description of behavior in applications such as viscoelastic media, thermal media, and electromagnetic systems [14–16]. Contrarily to the CFFD, previous fractional-order derivatives including the old caputo and other variant of fractional-order derivatives fitted more at modeling mechanical phenomena such as damage, plasticity, fatigue and fluid flow.

The well-posedness of a LEE model with the new Caputo-Fabrizio fractional derivative without singular kernel will be investigated. The existing literature in the field of fractional calculus indicates that there is still more to be done. New approaches in proving the existence and uniqueness of generalized fractional evolution equations have been studied. Our main focus is then on the investigation and evaluation of the new CFFD and its possible subsequent refinements on LEE. Consequently, two needs drive our interest in the proposed research. There are as follows:

- The need to demonstrate that the LEE modeled with the new CFFD is a well-posed problem and emphasize on a potential application in the field of applied sciences. Thus various techniques used in the literature will be investigated to prove the well-posedness of the LEE modeled with the new CFFD.
- The need to investigate the existence and uniqueness proofs of the recently developed Fractional derivative equations. Recent extensions of the original fractional derivative equations such Riemann-Liouville and Caputo fractional derivative equations to the newly developed CFFD will also be investigated.

The intended demonstration of the well-posedness for the problem formulated with the new CFFD in modelling LEE is a valuable contribution in the state-of-the-art of fractional calculus for the reason that, to the best our knowledge, such demonstration for LEE with CFFD has not been done. Our studies is only directed to the LEE though non-linear evolution equation can be object of future work.

1.4 Research aim and objectives

As already indicated in Section 1.1, the purpose of this study is to prove the well-posedness for the newly developed CFFD model applied to LEE by taking advantage of the fact that the integral in CFFD has a non-singular kernel. Once well-posedness established, advanced stability analysis of derived solutions can be effected.

A problem is well-posed if it satisfies the following three properties which are: (1) existence of a solution, (2) the existing solution is unique, and (3) the behavior's solution depends continuously on the data and parameters.These three properties summarises our research objectives as each of them will be investigated and demonstrated with respect to the LEE with CFFD.

1.5 Research Methodology

Various related topics such as applied functional analysis, Classical calculus, Fractional calculus, Partial differential equations, with linear evolution equations (LEE) in particular are investigated. A literature review on the fractional calculus and on the fractional differentiation in particular is included in the dissertation for good understanding of the history and development of this field of study. Strengths, weaknesses and limitations related to fractional derivatives and their applications are identified and elaborated. We have also reviewed the full literature on evolution equations and in particular the linear evolution equations. The evaluation methods applied to these evolution equations are presented and related theorems, existence and uniqueness of solutions established. The concepts of semigroup and solution operator theory have been considered for this matter, especially that the Semigroup theory provides the necessary and sufficient conditions to determine well-posedness. Hence we have made use of both methods and mathematical tools at our disposal to establish the well-posedness of the Caputo-Fabrizio time fractional derivative model as applied to linear evolution equations. Our investigation has used, as a departure point, the LEE defined as

$$\begin{aligned} {}_{0}^{CF}D_t^{\alpha}u(t) &= \mu u(t), \quad 0 < \alpha \leq 1, \ \ t > 0, \ \mu \in \mathbb{C} \\ u(0) &= f_0. \end{aligned} \tag{1.3}$$

where A is replaced by number $\mu \in \mathbb{C}$. Then the concepts of semigroup and solution operator have been used to extend and adapt Peano's idea to our models and establish conditions for existence and uniqueness of solutions. Furthermore, the relation between the solution operator, its resolvent and its generator have been also established.

1.6 Dissertation outline

This dissertation is structured as follows: Chapter 2 provides relevant literature review. A background on fractional calculus with related definitions, the Laplace transform as an evaluation method, and a functional approach to the analysis of LEE with linear operator on Banach spaces are also discussed.

In Chapter 3, the application of both the classical derivative and the old-Caputo time fractional derivative to Linear evolution equations are discussed. The Linear evolution equation as originally introduced in the form of ordinary differential equation (ODE) is analyzed in terms of the well-posedness property. Three cases of ODEs, depending on the type of operator whether is a scalar, matrix or a element of a Banach space, are revised for better understanding of the well-posedness property. This same property for LEE with CFD is discussed based on the concept of solution operators.

In Chapter 4, we present a mathematical analysis of the generalized linear evolution equations with the new Caputo-Fabrizio time fractional derivative. After highlighting the role played by the solution operator in proven the well-posedness, the Lipschitz condition and the Picard $\mathcal{T}$-stability are demonstrate that the LEE is well-posed. We further validate the extension of the LEE with CFFD by means of three applications, which are models for the kinetic physics, the heat diffusion, and the dispersion of shallow water waves.

Lastly, conclusion and future work are highlighted in Chapter 5.

Chapter 2

Literature review

"Mathematical Analysis is as extensive as nature herself."

Joseph Fourier

2.1 Introduction

The genesis of fractional calculus has been marked by the letter by l'Hôpital to Leibniz in which an important question with regards the order of the derivative and in particular what the derivative of order $\frac{1}{2}$ might signify. In [17] , de Oliveira referred to a prophetic answer by Leibniz who foresaw the beginning of the area that nowadays is named fractional calculus (FC). Katugampola in [6] mentioned that Abel (1823) was the first to apply the fractional calculus in the form of the semi-derivative in the solution to the tautocrone problem. This chapter thus highlights some important development of fractional calculus related to our study. The state-of-the-art definitions, theorems, properties and results used for the analysis of LEE with CFFD are provided. The literature on calculus of vectors valued functions, linear semigroup theory and solutions is also presented.

2.2 Background to Fractional Calculus

Fractional Calculus is a branch of mathematical analysis which deals with integro-differential operators and equations where integrals are of convolution type. Although the classical derivative represents the tangent of a function at the point by providing an important geometric interpretation of the first-order derivative, the fractional derivative as a excellent tool describes effectively the hereditary or memory properties of natural phenomena. The first definition of a derivative with an arbitrary order that used an integral representation was proposed by Fourier, then another definition based on differentiation of exponential function suggested by Liouville (1932) followed [6].

2.2.1 Definitions of fractional derivatives

Later, many different definitions of fractional derivatives were derived and presented in [17]. Among them, three basic and usually used in recent fractional derivative based studies are the Grünwald-Letnikov, Riemann-Liouville and Caputo fractional derivatives. These following definitions can also be found in [18].

1. **Grünwald-Letnikov Fractional Derivative**:

   The Grünwald-Letnikov fractional derivative with fractional order α if $x(t) \in C^m[0,t]$ is defined as follows:

$$ {}^{GL}D^{\alpha}_{0,t}x(t) = \sum_{k=0}^{m-1} \frac{x^k(0)t^{-a+k}}{\Gamma(-a+k+1)} + \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} x^{(m)}(\tau)d\tau \quad (2.1)$$

   with, $m-1 \leq \alpha < m \in Z^+$.

   This definition comes from a generalization of the classical derivative to a derivative with fractional order and derived as follows. Consider the first order derivative given by

$$D^{(1)}x(t) = \lim_{h\to 0} \frac{x(t)-x(t-h)}{h} \quad (2.2)$$

   the n^{th}–order derivative of $x(t)$ is

$$D^{(n)}x(t) = \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} x(t-kh) \quad (2.3)$$

   with $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ and $n \in Z$. Knowing the Gamma-function such that $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(n+1) = n!$ for $n \in Z^+$, we can write the following expression

$$(-1)^k \binom{n}{k} = \frac{-n(1-n)(2-n)...(k-n-1)}{k!} \quad (2.4)$$

   for $n \in Z^-$. By subtituting n by a non-integer α, we get

$$(-1)^k \binom{\alpha}{k} = \frac{-\alpha(1-\alpha)(2-\alpha)...(k-\alpha-1)}{k!} = \frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)}, \quad (2.5)$$

yielding to

$$D^{(\alpha)}x(t) = \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{k=0}^{\infty}\frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)}x(t-kh). \tag{2.6}$$

Thus

$$D^{(\alpha)}x(t) = \frac{1}{\Gamma(-\alpha)}\int_a^t (t-\tau)^{-\alpha-1}x(\tau)d\tau \tag{2.7}$$

known as the Grünwald-Letnikov definition. Consequently, if $x(t) \in C^m[0,t]$ and by applying the integration by parts, we get

$$D^{(\alpha)}x(t) = \sum_{k=0}^{m-1}\frac{x^k(0)t^{-a+k}}{\Gamma(-a+k+1)} + \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}x^{(m)}(\tau)d\tau, \tag{2.8}$$

which is the definition of the Grünwald-Letnikov Fractional Derivative.

2. **Riemann-Liouville Fractional Derivative**:
The Riemann-Liouville fractional derivative with fractional order α of $x(t)$ is defined as follows:

$$^{RL}D_{0,t}^{\alpha}x(t) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t (t-\tau)^{m-\alpha-1}x(\tau)d\tau \tag{2.9}$$

with, $m-1 \leq \alpha < m \in Z^+$.
This definition is derived as follows. Let $x : [0,\infty) \to R$ be a continuous function, then integral of order one is given by

$$(I_0^1 x)(t) = \int_0^t x(\tau)d\tau \tag{2.10}$$

and the one of second order by

$$(I_0^2 x)(t) = \int_0^t (\int_0^s x(t)dt)ds. \tag{2.11}$$

By using the Fubini theorem, we can write

$$(I_0^2 x)(t) = \int_0^t ((t-\tau)x(\tau)d\tau), \tag{2.12}$$

which by iteration we get

$$(I_0^n x)(t) = \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!}x(\tau)d\tau. \tag{2.13}$$

From (2.13), the integration of Riemann-Liouville of x is defined by

$$(I_0^n x)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}x(\tau)d\tau \tag{2.14}$$

where α is real. Based on the composition of functions, the Riemann-Liouville fractional derivative can be defined by

$$^{RL}D^{\alpha}_{0,t}x(t) = \frac{d^m}{dt^m}(I_0^{m-\alpha}x)(t). \tag{2.15}$$

Thus

$$\frac{d^m}{dt^m}(I_0^{m-\alpha}x)(t) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t (t-\tau)^{m-\alpha-1}x(\tau)d\tau \tag{2.16}$$

known as Riemann-Liouville fractional derivative.

3. **Caputo Fractional Derivative**:

   In the development of the theory of both fractional integration and derivation, as well in the related applications in pure mathematics, the Riemann-Liouville fractional derivative as defined in (2.9) played a important role. However, the solutions of problems in physics have required a revision of RLFD which is difficult to be interpreted physically. Hence the CFD was proposed.

   The Caputo fractional derivative with fractional order α of $x(t)$ is defined as follows:

$$^{C}D^{\alpha}_{0,t}x(t) = D^{-(m-\alpha)}_{0,t}\frac{d^m}{dt^m}x(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}x^m(\tau)d\tau \tag{2.17}$$

   with, $m-1 \leq \alpha < m \in Z^+$.

   This definition is more practical for analytic purpose than the Grünwald-Letnikov Fractional Derivative. Moreover, the initial conditions in the Caputo approach takes the same form as in the classical differential equations.

In the above definitions, the Gamma-function is used and plays a very important in FC. It is defined as follows:

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt \tag{2.18}$$

where $t^{z-1} = e^{(z-1)log(t)}$.

2.2.2 Properties of the fractional derivatives

A very well known property of fractional derivatives is the memory effect in describing natural phenomena. Integer-order derivatives are quite limited at this regard. The Riemann-Liouville fractional derivatives need initial conditions expressed in terms of initial values of fractional derivatives of the unknown functions whereas the initial conditions for Caputo fractional derivatives are expressed in terms of initial values of integer order derivatives [15]. Since the physical meaning of initial conditions of fractional derivatives is unclear or even non existent, the Caputo fractional derivatives are more solicited for applications in applied sciences [19].

In [4, 5, 18], a relationship between CFD and RLFD for $1 < \alpha \in (m-1, m)$, $m \in Z^+$ is given by:

$$ {}^{C}D^{\alpha}_{0,t}x(t) = {}^{RL}D^{\alpha}_{0,t}x(t) - \sum_{k=0}^{m-1} \frac{t^k x^k(0)}{k!}. \tag{2.19} $$

From the definition of CFD, the function $x(t)$ requires to be differentiable in the integrand implying that less functions can be derived in the Caputo sense than in the Riemann-Liouville sense. Another particularity of CFD is that it deals only with differentiable functions. In addition, the RLFD of a constant function is not zero whereas the one from the CFD is zero. Comparing the CFD and RLFD, one could mention that the CFD is more suitable in solving problems described by FDEs [17].

2.3 Derivative with non-singular kernel: Caputo-Fabrizio Fractional derivative

Among existing fractional derivatives, the most commonly used are the RLFD and the CFD also known as the old Caputo derivative. The new CFFD without singular kernel is simply an extension of the old CFD where the kernel of the integral is reformulated. The definition of the new CFFD presented in this section is extracted from [14].

For $m = 1$ and a being an initial value other than 0 such that $a \in (-\infty, t]$, the equation (2.17) then becomes

$$ {}^{C}D^{\alpha}_{a,t}x(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} \dot{x}(\tau) d\tau \tag{2.20} $$

with $\alpha \in [0, 1]$, $x \in H^1(0, b)$.

By changing the kernel $(t-\tau)^{-\alpha}$ with the function $exp(-\frac{\alpha}{1-\alpha}t)$ and $\frac{1}{\Gamma(1-\alpha)}$ with $\frac{M(\alpha)}{1-\alpha}$, the CFFD is defined by

$$ {}^{CF}D^{\alpha}_{t}x(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t \dot{x}(\tau) exp[-\frac{\alpha(t-\tau)}{1-\alpha}] d\tau, \tag{2.21} $$

where $\alpha \in [0, 1]$, $a \in [-\infty, t)$, $x \in H^1(a, b)$ and $M(\alpha)$ is a normalisation constant.

When x does not belong to $H^1(a,b)$, the equation (2.21) is re-formulated for $x \in L^1(-\infty, b)$ and for $\alpha \in [0,1]$, and gives

$$^{CF}D_t^\alpha x(t) = \frac{\alpha M(\alpha)}{(1-\alpha)} \int_{-\infty}^{t} (x(t) - x(\tau) exp[-\frac{(t-\tau)}{1-\alpha}] d\tau. \tag{2.22}$$

It is clear that, compared to the old Caputo, the kernel does not have singularity at $t = \tau$ as is the case with the old Caputo. Moreover, the CFFD, if we pose $\sigma = \frac{1-\alpha}{\alpha} \in [0,\infty]$ and $\alpha = \frac{1}{1+\sigma} \in [0,1]$, can assume the form

$$D_t^\sigma x(t) = \frac{N(\sigma)}{\sigma} \int_a^t \dot{x}(t) exp[-\frac{(t-\tau)}{\sigma}] d\tau \tag{2.23}$$

where $N(\sigma)$ is the corresponding normalization term of $M(\alpha)$ such that $N(0) = N(\infty) = 1$. Since,

$$\lim_{\sigma \to 0} \frac{1}{\sigma} exp(-\frac{t-\tau}{\sigma}) = \delta(t-\tau). \tag{2.24}$$

and for $\alpha \to 1$, we have $\sigma \to 0$. This leads to the following remarks.

Remark 2.1. The first order derivation of CFFD is similar to the one of the classical derivative:

$$\lim_{\alpha \to 1} {}^{CF}D_t^\alpha x(t) = \lim_{\sigma \to 0} D_t^\sigma x(t) = \dot{x}(t). \tag{2.25}$$

Remark 2.2. The first order derivation of CFFD is similar to the one of the classical derivative:

$$\lim_{\alpha \to 0} {}^{CF}D_t^\alpha x(t) = \lim_{\sigma \to +\infty} D_t^\sigma x(t) = x(t) - x(a). \tag{2.26}$$

In their paper, Losada and Nieto re-formulated the CFFD to the simple form defined below.

Definition 2.1[20] Let $0 < \alpha < 1$. The CFFD derivative of order α of a function x is defined by

$$^{CF}D_t^\alpha x(t) = \frac{1}{(1-\alpha)} \int_0^t exp[-\frac{\alpha(t-\tau)}{1-\alpha}] \dot{x}(\tau) d\tau \tag{2.27}$$

for $t \geq 0$. This reformulation came from their suggested anti-derivative associated to ${}^{CF}D_t^\alpha x(t) = u(t)$ for $t \geq 0$ and defined by

$$^{CF}I_t^\alpha x(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} u(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t u(\tau) d\tau, \quad t \geq 0, \tag{2.28}$$

which is enforced by

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} = 1, \tag{2.29}$$

resulting from the mean between the function x and its integral of order one where

$$M(\alpha) = \frac{2}{(2-\alpha)}, \quad 0 \leq \alpha \leq 1. \tag{2.30}$$

This derivative with non-singular kernel would better describe the evolution of systems with hereditary/memory effect. Therefore the definition by Losada and Nieto are used to proof well-posedness of linear evolution equation extended with CFFD.

2.4 Recent research in Fractional derivatives with non-singular kernel

Yang et *al* in [21] formulated a fractional derivative operator without singular kernel, which is an analogue of the well-known Riemann-Liouville fractional derivative with singular kernel. The new fractional derivative as per [21] is

$$D_a^{\gamma}T(x) = \frac{R(\gamma)}{1-\gamma}\frac{d}{dx}\int_a^x exp(-\frac{\gamma}{1-\gamma}(x-\lambda))T(\lambda)d\lambda \tag{2.31}$$

where $a \leq x$, γ $(0 < \gamma < 1)$ is a real number and R is a normalisation number as per the new CFFD definition, satisfying $R(0) = R(1) = 1$.

Atangana and Baleanu published an article [22] in which they proposed a new fractional derivative with a kernel that is non-local and non-singular. They introduced two versions, i.e:

$${}_b^{ABC}D_t^{\alpha}f(t) = \frac{\beta(\alpha)}{(1-\alpha)}\int_b^t \acute{f}(x)E_{\alpha}[-\frac{(t-x)^{\alpha}}{1-\alpha}]dx, \tag{2.32}$$

which is called the Atangana-Baleanu(AB) fractional derivative in the Caputo sense, since it is an extension of the new CFFD and

$${}_b^{ABR}D_t^{\alpha}f(t) = \frac{\beta(\alpha)}{(1-\alpha)}\frac{d}{dt}\int_b^t f(x)E_{\alpha}[-\frac{(t-x)^{\alpha}}{1-\alpha}]dx, \tag{2.33}$$

which is called the Atangana-Baleanu (AB) fractional derivative in the Riemann sense, since it is an extension of the Riemann-Liouville fractional derivative.

These two authors indicate in [22] that the first definition "will be helpful to discuss real world problems and it also will have great advantage when using the Laplace transform to solve some physical problems with initial condition." As they have shown for the new CFFD as well, with the ${}_b^{ABC}D_t^{\alpha}$ definition the original function is not recovered when α

is zero, except at the origin where the function vanishes. The second definition above, the ${}_{b}^{ABR}D_{t}^{\alpha}$ definition is addressing this challenge.

According to [22], the definitions is believed to "be very useful in describing many complex problems in thermal sciences." The authors also assert that kernel's non-locality allow better description of memory within the structure and media with different scales [22]. The non-locality of this definition stems from the use of the generalized Mittag-Leffler function, which is considered non-local, in the formulations of the fractional derivative above. These definitions employed the fact that the M-L function is a generalisation of the exponential function [23] and used it to re-formulate the new CFFD and the R-L fractional derivative.

In the article by Lv, Wang and Wei of 2011 [24] they use the Schaefer fixed point theorem to establish the existence and uniqueness Fractional differential equations bondary value problem.
Alqahtani in his paper, [25] employs a fixed point theorem to prove the existence and uniqueness of the non-linear Nagumo equation. There is also the article by Hristov [26] where he uses the Caputo-Fabrizio time fractional derivative to model the transient heat diffusion for homogeneous rigid heat conductors.

The use of the old Caputo fractional derivative to model LEE and the establishment of the well-posedness of the model have been done before, notably by Bazhlekova, with her article [7] outlining some of the research in approximation properties.

2.5 Evaluation methods for Fractional differential equations

Various methods for evaluating FDEs including ODEs exist in the literature [5],[27] and is explored in the dissertation. To name few, we have [27]: (1) the interative method which is effective in solving only simple FDEs with real order, (2) the Laplace transform method which is suitable for evaluating the FDE based IVPs. This method in association with the Mittag-Leffler function defined as $E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$ has been used in a widespread way to formulate solutions of FDEs. In addition, Laplace transform overcomes some challenges faced by the iterative method and it has been a popular analytical tool in signal processing and engineering field in general. In spite of Adomian decomposition method, Homotopy analysis method, explicit numerical method and the Variational iterative method, Laplace transform is considered here and discussed with respect to fractional derivative types in more details in this section. The Laplace transform of a function $x(t)$ denoted by $X(s)$ is defined by

$$X(s) = \mathcal{L}(x(t))(s) = \int_0^\infty e^{-st}x(t)dt = \lim_{\tau\to\infty}\int_0^\tau e^{-st}x(t)dt, \tag{2.34}$$

whenever the limit exists as a finite number.

The Mittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ka+1)}, \tag{2.35}$$

where $\alpha > 0$ and $z \in C$. And the two parameter Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ka+\beta)}, \tag{2.36}$$

where $\alpha > 0$, $\beta > 0$ and $z \in C$.

2.5.1 Laplace transform of GLFD

Let assume that the function x admits a Laplace transform $X(s)$, then according to (2.1) we have for $0 < \alpha < 1$,

$${}^{GL}D^\alpha_{0,t}x(t) = \frac{x(0)t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\dot{x}(\tau)d\tau, \tag{2.37}$$

whose the corresponding Laplace transform is given by

$$\mathcal{L}\{{}^{GL}D^\alpha_{0,t}x(t), s\} = \frac{x(0)}{s^{1-\alpha}} + \frac{1}{s^{1-\alpha}}(sX(s) - x(0)) = s^\alpha X(s). \tag{2.38}$$

In addition, for application purposes, it is important to mention that the Laplace transform of GLFD has a sense in classical case where $0 < \alpha < 1$. Otherwise for $\alpha > 1$ the transform is treated as distributions [28].

2.5.2 Laplace transform of RLFD

The fractional integral of Riemann-Liouville as defined in (2.14) can be expressed as a convolution product $h(t) = t^{\alpha-1}$ and $x(t)$ as follows.

$$(I_0^n x)(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}x(\tau)d\tau = \frac{(t)^{\alpha-1}}{\Gamma(\alpha)} * x(t). \tag{2.39}$$

The Laplace transform of $h(t)$ is given by

$$H(s) = \mathcal{L}\{t^{\alpha-1}, s\} = \Gamma(\alpha)s^{-\alpha}. \tag{2.40}$$

So by applying the convolution property for Laplace transform, the Laplace transform of the fractional integral of Riemann-Liouville becomes

$$\mathcal{L}\{^{RL}D^{\alpha}_{0,t}x(t), s\} = s^{-\alpha}X(s). \tag{2.41}$$

In order to determine the Laplace transform of the RLFD of the function $x(t)$, let

$$^{RL}D^{\alpha}_{0,t}x(t) = h^{(m)}(t), \tag{2.42}$$

yielding to

$$h(t) = {}^{RL}D^{-(m-\alpha)}_{0,t}x(t)\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}x(\tau)d\tau, \quad m-1 \leq \alpha < m. \tag{2.43}$$

By applying the Laplace transform on (2.42), we get

$$\mathcal{L}\{^{RL}D^{\alpha}_{0,t}x(t), s\} = s^{\alpha}H(s) - \sum_{k=0}^{m-1} s^k h^{(m-k-1)}(0), \tag{2.44}$$

where

$$H(s) = s^{-(m-\alpha)}X(s). \tag{2.45}$$

Based of the definition of RLFD, we get

$$h^{(m-k-1)}(t) = \frac{d^{m-k-1}}{dt^{m-k-1}}{}^{RL}D^{-(m-\alpha)}_{0,t}x(t) = {}^{RL}D^{\alpha-k-1)}_{0,t}x(t). \tag{2.46}$$

By substituting (2.45) and (2.46) in (2.44), the final expression of the Laplace transform of the RLFD of the function x is

$$\mathcal{L}\{^{RL}D^{\alpha}_{0,t}x(t), s\} = s^{\alpha}X(s) - \sum_{k=0}^{m-1} s^k[^{RL}D^{\alpha-k-1)}_{0,t}x(t)]_{t=0}, \quad m-1 \leq \alpha < m. \tag{2.47}$$

2.5.3 Laplace transform of CFD

The Laplace transform of the CFD for a function $x \in [a, \infty]$, $m-1 \leq \alpha < m$ and $a \leq 0$ is defined as follows.

In case where $a = 0$, we apply directly the Laplace definition to (2.17), which after integration technique such as change of variable, gives

$$\mathcal{L}\{{}^{C}D_{0,t}^{\alpha}x(t), s\} = \mathcal{L}\{\frac{t^{m-1-\alpha}}{\Gamma(m-\alpha)}, s\}\mathcal{L}\{x^{(m)}(t), s\} = s^{\alpha-m}\mathcal{L}\{x^{(m)}(t), s\}. \tag{2.48}$$

But $\mathcal{L}\{x^{(m)}(t), s\} = s^{n}\mathcal{L}\{x(t), s\} - s^{m-1}x(0) - s^{m-2}\dot{x}(0) - ... - x^{(m-1)}(0)$, hence

$$\mathcal{L}\{{}^{C}D_{0,t}^{\alpha}x(t), s\} = s^{\alpha}\mathcal{L}\{x(t), s\} - s^{\alpha-1}x(0) - s^{\alpha-2}\dot{x}(0) - ... - s^{\alpha-m}x^{(m-1)}(0). \tag{2.49}$$

In the general case, the Laplace transform of CFD of a function x is then defined by

$$\mathcal{L}\{{}^{C}D_{0,t}^{\alpha}x(t), s\} = s^{\alpha} - e^{-as}[s^{\alpha}\mathcal{L}\{x(t), s\} - s^{\alpha-1}x(a) - s^{\alpha-2}\dot{x}(a) - ... - s^{\alpha-m}x^{(m-1)}(a)]. \tag{2.50}$$

2.5.4 Laplace transform of CFFD

The Laplace transform, same as in the case of ODEs, plays an important role for analysis of CFFD derivatives. The corresponding Laplace transform of CFFD of a function x for $0 < \alpha < 1$ is defined by

$$\mathcal{L}\{{}^{CF}D_{0,t}^{\alpha}x(t), s\} = \frac{(2-\alpha)M(\alpha)}{2(s+\alpha(1-s))}[s\mathcal{L}\{x(t), s\} - x(0)], \quad s > 0. \tag{2.51}$$

The Laplace transform of the generalized CFFD is given in [14] by

$$\mathcal{L}\{{}^{CF}D_{0,t}^{\alpha+m}x(t), s\} = \frac{1}{s+\alpha(1-s)}[s^{\alpha+m}\mathcal{L}\{x(t), s\} - s^{m}x(0) - s^{m-1}\dot{x}(0) - ... - x^{(m)}(0)]. \tag{2.52}$$

2.6 Functional calculus of linear operators on Banach Spaces

In this section, we highlight relevant techniques on vector valued functions and introduce briefly concepts of functional analysis used in the dissertation. This includes definitions and theorems related to spaces, and linear operators.

2.6.1 Bounded linear operators

A vector-valued function can be referred to as a function that takes values in Banach spaces. It is laso known as a vector function defined as a mathematical function of one or more variables whose range is a set of multidimensional or infinite-dimensional vectors.This function could take in as input a scalar or a vector to product a unique vector in a Banach space. Moreover such function is also called an operator-valued function if it belongs to a Banach space which is a space of bounded linear operators.

Definition 2.1. ***(Linearity)*** Let $(X, \|.\|)$ and $(Y, \|.\|)$ be two Banach spaces over the same field $\mathbb{K}$. A mapping $A : D(A) \subset X \to Y$ satisfying

$$A\left(k_1 x + k_2 y\right) = k_1 Ax + k_2 Ay, \tag{2.53}$$

for all $x, y \in D(A)$ and $k_1, k_2 \in \mathbb{K}$, is called linear operator.

Definition 2.2. ***(Bounded or Continuity)*** A linear operator $A : X \to Y$ satisfying the following property:
there exists $k \leq 0$ such that

$$\|Ax\| \leq kAx \tag{2.54}$$

for all $x \in X$ and k a constant, is called a bounded or continuous linear operator.

Other properties such as differentiality and integrability of vector-valued functions, which are comparable to the scalar case, are presented here.

Definition 2.3. ***(Differentiability)*** Consider X, a Banach space with norm $\|\cdot\|_X$ and a function f, an X-valued function of $t \in [0, \infty)$. Then f is strongly differentiable at $t \in [0, \infty)$, with a strong pointwise derivative $\dfrac{df(t)}{dt}$ of f, if

$$\frac{df(t)}{dt} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}, \tag{2.55}$$

or

$$\lim_{h \to 0} \left\| \frac{f(t+h) - f(t)}{h} - \frac{df(t)}{dt} \right\|_X = 0, \tag{2.56}$$

provided that these limits exist.

Definition 2.4. ***(Integrability)*** Let X be a Banach space with norm $\|\cdot\|_X$ and let the function f be an X-valued function of $t \in [0, \infty)$. Then $f :\in [0, \infty) \to X$, which is strongly measurable, is integrable if there exists a sequence of simple functions such that $f_n \to f(t)$ pointwise in $[0, \infty)$ and

$$\lim_{n \to \infty} \int_0^\infty \|f(t) - f_n(t)\|_X dt = 0. \tag{2.57}$$

Equation (2.58) yields the defintion of the integral of f as

$$\int_0^\infty f(t)dt = \lim_{n \to \infty} \int_0^\infty \|f_n(t)\|_X dt, \tag{2.58}$$

where the limit strongly exists in X. See [29, Theorem 10.24]. Also if the integral f is independent of the sequence $\{f_n\}$, then following inequality

$$\left\| \int_0^\infty f(t)dt \right\|_X = \int_0^\infty \|f(t)\|_X \, dt, \tag{2.59}$$

holds.

In addition, if $A : X \to Y$ is a bounded linear operator between the Banach spaces X, Y, and $f : [0, \infty) \to X$ integrable, then $Af : [0, \infty) \to Y$ is also integrable and

$$A \left(\int_0^\infty f(t)dt \right) = \int_0^\infty Af(t)dt. \tag{2.60}$$

Theorem 2.5. *Equation (2.60) generally holds whenever $A : D(A) \subset X \to Y$ is a closed linear operator, $f : [0, \infty) \to D(A)$ a strong continuous vector-valued function on $[0, \infty)$ whose $\int_0^\infty f(t)dt \in D(A)$, and Af is strongly continuous on $[0, \infty)$.*

Proof. [30, Theorem 3.3.2]. □

Various types of bounded linear operators are derived based on the transformation performed on the operator A. Few types of bounded linear operators are listed below.

- If the adjoint A^* of A exists, then A^* is also a bounded linear operator.
- If the operator A is said to be compact, then A is referred to as a compact operator.
- If the operator A is invertible, then its inverse A^{-1} is also a bounded linear operator called inverse operator.

Nevertheless, linear operators can present some discontinuity at certain points of its domain of definition. Such operators are called unbounded linear operators and they form part of a important class of operators on the Banach space.

2.6.2 Unbounded linear operators

Definition 2.6. Let $A : X \to Y$ be an linear operator. The operator A is called unbounded linear operator if A is not continuous.

A typical example of unbounded linear operator, discussed in [31], is presented as follows. Let $X = Y = L^2(\mathbb{R})$ and consider the 1D-Laplace operator defined by

$$Au = -u'' \quad \text{and} \quad D(A) = H^2(\mathbb{R}) \tag{2.61}$$

for all $u \in H^2(\mathbb{R})$.

Let consider the sequence of functions $\Phi_n(t) = e^{-n|t|}$ for $n = 1, 2, 3, \ldots$, such that $\Phi_n(t) \in D(A) = H^2(\mathbb{R})$. Then

$$\|\Phi_n(t)\|_2^2 = \int_{-\infty}^{+\infty} e^{-2n|t|} dt = \frac{1}{n}$$

and

$$\|A\Phi_n(t)\|_2^2 = \int_{-\infty}^{+\infty} n^4 e^{-2n|t|} dt = n^3.$$

Thus, $\frac{\|A\Phi_n(t)\|_2}{\|\Phi_n(t)\|_2} = n \to \infty$ when n tends to ∞. Hence the operator A is an unbounded linear operator on $L^2(\mathbb{R})$.

2.6.3 Positive operators

The field of functional analysis also deals with Banach space $(X, \|.\|)$ such that

$$|x| \leq |y| \implies \|x\| \leq \|y\| \tag{2.62}$$

for all $x, y \in X$ and where $|x| := x \vee -x$. Such Banach space is called Banach lattice, which is a Riesz space with the norm $\|.\|$.

Definition 2.7. A Banach lattice is a real Banach space X endowed with an ordering $\leq$ such that $(X, \leq)$ is a vector lattice and the norm on X is a lattice norm.

Definition 2.8. Let X and Y be two Banach lattices Y. The linear operator $A : X \to Y$ is called positive, denoted by $A \geq 0$, if $Ax \geq 0$ for any $x \geq 0$.

Proposition 2.9. *[32] If a linear operator A is positive, then*

$$\|A\| = \sup_{x \geq 0,\, \|x\| \leq 1} \|Ax\|. \tag{2.63}$$

Theorem 2.10. *Assume that X is a weakly sequentially complete Banach lattice. If $(x_n)_{n\in\mathbb{N}}$ is increasing and $(\|x_n\|)_{n\in\mathbb{N}}$ is bounded, then there is $x \in X$ such that*

$$\lim_{n\to\infty} x_n = x \tag{2.64}$$

in X. In other words, weakly sequentially complete and, in particular, reflexive Banach lattices are KB-spaces.

Proof. [32]. □

2.6.4 Linear semigroups

In this section, important definitions leading to the solutions of a Cauchy problem is provided.

Definition 2.11. Given a Banach space X and a linear operator $\mathcal{A}$ with domain $D(\mathcal{A})$ and range Im$\mathcal{A}$ contained in X and also given an element $u_0 \in X$, find a function $u(t) = u(t, u_0)$ such that

(1) $u(t)$ is continuous on $[0, \infty)$ and continuously differentiable on $(0, \infty)$,

(2) for each $t > 0$, $u(t) \in D(\mathcal{A})$ and

$$\frac{du}{dt}(t) = \mathcal{A}u(t), \quad t > 0, \tag{2.65}$$

(3)

$$\lim_{t \to 0} u(t) = u_0 \tag{2.66}$$

in the norm of X. A function satisfying all conditions above is called the classical (or strict) solution of (2.65), (2.66).

Definition 2.12. A family $(S(t))_{t \geq 0}$ of bounded linear operators on X is called a C_0-semigroup, or a strongly continuous semigroup, if (i) $S(0) = I$; (ii) $S(t + s) = S(t)S(s)$ for all $t, s \geq 0$; (iii) $\lim_{t \to 0^+} S(t)x = x$ for any $x \in X$. A linear operator A is called the (infinitesimal) generator of $(S(t))_{t \geq 0}$ if

$$Ax = \lim_{h \to 0^+} \frac{S(h)x - x}{h}, \tag{2.67}$$

where the domain of A, $D(A)$, is defined as the set of all $x \in X$ for which this limit exists. Typically the semigroup generated by A is denoted by $(S_A(t))_{t \geq 0}$.

2.6.5 Solution operators

Definitions related to solution operators from [8] are also provided here. Let A be a closed linear operator densely defined in a Banach space X. If we let $\alpha > 0$ and $n \in \mathbb{N}$ such that $n - 1 < \alpha \leq n$ and $x \in X$, we define the Cauchy-problem for the fractional evolution equation of order α, (FE_α) as follows [8]:

$$D_t^\alpha u(t) = Au(t), \quad u(0) = x, \quad u^{(k)}(0) = 0, \quad k = 1, \ldots, n - 1 \qquad (FE_\alpha) \tag{2.68}$$

$\mathcal{B}(X)$ is the space of all bounded linear operators in X.

Definition 2.13. A family $(S(t))_{t\geq 0} \subset \mathcal{B}(X)$ is called a solution operator for the functional equation of fractional order, (FE_α) if the following conditions are satisfied:

(i) $S_\alpha(t)$ is strongly continuous for $t \geq 0$ and $S_\alpha(0) = I$;

(ii) $S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A), t \geq 0$.;

(iii) $S_\alpha(t)x$ is a solution of (FE_α) for all $x \in D(A), t \geq 0$.

It should be noted that if (FE_α) has a solution operator $S_\alpha(t)$, then the corresponding problem with initial conditions in the general form $u^{(k)}(0) = x_k, k = 0, \ldots, n-1$ is uniquely solvable with the solution $u(t) = \sum_{k=0}^{n-1} J_t^k S_\alpha(t)x(k)$, provided that $x_k \in D(A)$ and J_t^k being the fractional integral of order k. Hence our focus point is on (FE_α).

Bazhlekova in [8] provided the following definition relating to the solution operator $S_\alpha(t)$.

Definition 2.14. The solution operator $S_\alpha(t)$ is called exponentially bounded, if we have constants, $M \geq 1$ and $\omega \geq 0$ such that:

$$\|S_\alpha(t)\| \leq Me^{\omega t}, \quad t \geq 0. \tag{2.69}$$

The operator A belongs to $\mathcal{C}^\alpha(M,\omega)$, if the problem (FE_α) has a solution operator $S_\alpha(t)$ satisfying (2.69). We denote $\mathcal{C}^\alpha(\omega) = \bigcup\{\mathcal{C}^\alpha(M,\omega); M \geq 1\}$, $\mathcal{C}^\alpha = \bigcup\{\mathcal{C}^\alpha(\omega); \omega \geq 0\}$. With these notations $\mathcal{C}^1$ and $\mathcal{C}^2$ are sets of all infinitesimal generators of C_0-semigroups and Cosine families respectively.

2.7 Summary

The background on fractional calculus was introduced in Section 2.2 and related definitions and properties mentioned. In Section 2.3, the state-of-the art definitons of the new CFFD as a fractional derivative with non-singular kernel was presented. Other non-singular fractional derivatives were also highlighted in Section 2.4. Further, the Laplace transform of various fractional derivatives was presented as an evaluation method for solving FDE in Section 2.5. Lastly, based of the fact LEE is presented as a abstract Cauchy problem, the concept of linear semigroups for ODE and solution operators for FDE was introduced in Section 2.6 to deal LEE with linear operators on Banach spaces.

Chapter 3

Analysis of ordinary and Caputo-fractional linear evolution equations

"In the Calculus of feelings, you never really know how one person's absence will affect you more than another's."

Gayle Forman

3.1 Introduction

In mathematics, evolution equations are built by balancing the change of the system in time against its 'spatial' behavior [33]. Often these equations are expressed in terms of differential or integro-differential equations and analyzed in the calculus sense. By analysis, we imply finding the exact solution of a given evolution equation in terms of elementary functions, which is not in many cases a straightforward exercise. Other approaches to model validation without attempting to find solutions, have for goals to determine whether the solutions exist, whether they are unique, and how stable these solutions are with respect to associated parameters. Meeting these three criteria which are (1) existence, (2) uniqueness and (3) stability demonstrates the well-posedness of a given model. As a system evolves from one state to another, its evolution is described by a family of operators, forming a semigroup parameterized by time, that maps an initial state of the system to subsequent states. However, the theory of linear semigroup is well advanced [34–37] and provides necessary and sufficient conditions to determine the well-posedness of a problem [38]. Hence, in this chapter we present the theory of semigroup as it is applied to ODEs for better understanding of the concept. In addition, Solutions operators, which is an extension of semigroup to fractional differential equations, are also presented here for generalization purpose and their applications to LEE with CFD discussed.

3.2 Analysis of ordinary evolution equations

An evolution equation is the interpretation of differential law of the development in time of a system. Such equation provides possibilities to build solutions from a given initial condition which is nothing else than the initial state of the system. Different type of evolution equations, as initially introduced, includes ordinary differential equations. To conduct our analysis, we consider the following ODE definition for LEE also known a IVP or Cauchy problem:

$$\frac{d}{dt}x(t) = Ax(t), \quad t > 0, \lim_{t \to 0} x(t) = x_0 \tag{3.1}$$

where $x(t)$ is solution and describes the state of the system at time t which changes in time at the rate given by the operator A.

Three variants of the above defintion are considered here based on the nature of their linear operator denoted by A to demonstrate the importance of semigroup.

3.2.1 Scalar ordinary evolution equation

Here, the operator A is a scalar denoted by a. The corresponding problem is the simplest ODE and defined by

$$\frac{d}{dt}x(t) = ax(t), \quad t > 0, \quad a \in R, \quad x(0) = x_0 \tag{3.2}$$

The solution to (3.2) is $x(t) = e^{at}x_0$ for $t \geq 0$. Since the operator a is a real parameter, then the following behavior can be observed with respect to a:

- For $a < 0$ and when $t \to +\infty$ then $x(t) \to 0$ implying that the solution $x(t)$ is asymptotically stable.
- For $a = 0$ the solution $x(t)$ is stable but not asymptotically.
- For $a > 0$ and when $t \to +\infty$ then $x(t) \to +\infty$ implying that the solution is unstable.

Furthermore, let S operate on x as follows:

$$S(t) : x(s) \longmapsto S(t)x(s) = x(t+s). \tag{3.3}$$

From (3.7), $x(t+s)$ is solution at time $t+s$, which can be read as the operator $S(t+s)$ acting on $x(0)$. Thus $x(t+s) = S(t+s)x_0$. Considering the evolution effect of EE, the solution or state of the system at time $t+s$ can also achieved by either mapping directly the initial state x_0 to the state at time $t+s$ or by allowing the state to evolve over s time units. The operator $S(.)$ is acting like a transition operator [39].

So,

$$S(t+s) = S(t)S(s), \quad t, s > 0, \tag{3.4}$$

corresponds to a semigroup property which exponential function can satisfy. Consequently here, $S(t) = e^{at}$ and $\{S(t)\}_{t\geq 0}$ is a family of bounded linear map from R into R also fulfilling $S(0) = 1$. The property $S(0) = 1$ means no transition at time 0. A unique family $\{S(t)\}_{t\geq 0}$ of semigroup is attached to every ODE (3.2) and each family $\{S(t)\}_{t\geq 0}$ of semigroup is attached to an ODE as defined in (3.2) where the generator a is found by

$$\lim_{h\to 0^+} \frac{S(t) - 1}{t} = a. \tag{3.5}$$

Hence it is worth mentioning that the well-posedness property is closely connected to the family $\{S(t)\}_{t\geq 0}$ of uniformly continuous semigroup of bounded linear operators with the scalar a as a generator.

3.2.2 System of ordinary evolution equations

Here the operator A is a matrix denoted by A. the corresponding problem is a system of ODEs defined by

$$\frac{d}{dt}x(t) = Ax(t), \quad x(0) = x_0, \quad A \in R^N \tag{3.6}$$

where for each $t \geq 0$, the corresponding unique solution is $x(t) \in R^N$ and A is a MxM real matrix. This solution can be written as $x(t) = e^{At}x_0$ with $e^{At} := \sum_{i=0}^{\infty} \frac{A^i t^i}{i!}$ and $A^0 = I$. I is a MxM identity matrix.

Since the commutation property is assured with squared MxM matrices, therefore the semigroup property

$$S(t+s) = e^{(t+s)A} = e^{tA}e^{sA} = S(t)S(s) \tag{3.7}$$

holds.

Furthermore, the family $\{S(t)\}_{t\geq 0}$ forms a semigroup of bounded linear operator from R^M into R^M. This family can be referred to as an uniformly continuous semigroup for

$$\lim_{t\to 0^+} S(t) = I, \tag{3.8}$$

with the corresponding generator

$$A = \lim_{t\to 0^+} \frac{S(t) - I}{t}. \tag{3.9}$$

Hence it is worth mentioning that the well-posedness property is also closely connected to the family $\{S(t)\}_{t\geq 0}$ of uniformly continuous semigroup of bounded linear operators with the matrix A as a generator.

In addition, if A is a real symmetric matrix, then it is diagonalizable. Thus the solution of (3.6) can be expressed as

$$x(t) = S(t)x_0 = \sum_{i=1}^{M} e^{\lambda_i}(x_0, \phi_i)\phi_i, \tag{3.10}$$

where ϕ_i and λ_i, $i = 1, ..., M$ are eignevectors and eigenvalues respectively. Moreover, the stability of the system here will depend on the sign of the eigenvalues.

3.2.3 Ordinary evolution equation in Banach spaces

Here A is a bounded linear operator on the Banach space X with norm $||.||$, that is $A \in BL(X)$. The corresponding evolution equation is then defined by

$$\frac{d}{dt}x(t) = Ax(t), t \geq 0 \quad x(0) = x_0, \quad A \in X, \tag{3.11}$$

where the solution is $x(t) = e^{tA}x_0$. With $e^{At} := \sum_{i=0}^{\infty} \frac{A^i t^i}{i!}$ and $S(t) = e^{tA}$, then the family $\{S(t)\}_{t\geq 0}$ forms a uniformly continuous semigroup of bounded linear operators on the Banach space X.

Furthermore, if X is a Hilbert space with inner product $(.,.)$, and A a selfadjoint and compact linear operator on X, then a countable number of real eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ can be derived. For the corresponding normalized eigenvectors $\{\phi_i\}_{i=1}^{\infty}$ forming an orthonormal basis of the Banach space X, then the solution is

$$x(t) = S(t)x_0 = \sum_{i=1}^{\infty} e^{\lambda_i}(x_0, \phi_i)\phi_i. \tag{3.12}$$

In addition, the solution is stable if at least one eigenvalue is zero with all the others negative, and for the induced norm $||.||$ on X such that

$$||S(t)x_0|| = ||x(t)|| \leq \sum_{i=1}^{\infty} ||(x_0, \phi_i)\phi_i||. \tag{3.13}$$

Finally, we conclude that both the well-posedness or solvability of (3.11) is also closely connected with the existence of a family $\{S(t)\}_{t\geq 0}$ of uniformly continuous semigroup of bounded linear operators with the $A \in BL(X)$ as a generator.

3.2.4 More on semigroup operators

Consider the linear homogeneous evolution equation

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t), \quad t > 0 \\ x(0) &= x_0, \quad A \in X, \end{aligned} \tag{3.14}$$

where A is linear but not necessarily bounded on X with domain $D(A) \subset X$.

The cases discussed in Section 3.2 gives us a good understanding of the concept of semigroup operator as applied to LEEs. We now formally highlght related definitions, theorems, remarks and properties. Here we assume that X is a Banach space with norm $\|.\|$.

Definition 3.1. A family $\{S(t)\}_{t\geq 0}$ of bounded linear operators on X is called a semigroup on X, if it satisfies

1. $S(0) = I$,
2. $S(t+s) = S(t)\,S(s)$, $\forall t, s \geq 0$
   if moreover,
3. $\lim_{t\to 0^+} \|S(t)x - x\|_X = 0$, for all any $x \in X$, then $\{S(t)\}_{t\geq 0}$ is a strongly continuous semigroup called C_0-semigroup.

Definition 3.2. A linear operator A defined by

$$Ax = \lim_{t \to 0^+} \frac{S(t)x - x}{t},$$

whose domain of definiton is

$$D(A) = \left\{ x \in X, \lim_{t \to 0^+} \frac{S(t)\,x - x}{t}, \text{exists} \right\},$$

is called infinitesimal generator of the family of semigroups $\{S(t)\}_{t \geq 0}$.

Definition 3.3. A family $\{S(t)\}_{t \geq 0}$ is called uniformly continuous with respect to the norm $\|.\|$ associated with X, if

$$\lim_{t \to 0^+} \|S(t) - I\|_X = 0$$

Definition 3.4. A family $\{S(t)\}_{t \geq 0}$ is called strongly continuous with respect to the norm $\|.\|$ associated with X, if

$$\lim_{t \to 0^+} \|S(t)x - x\|_X = 0$$

So far, two types of linear semigroups are introduced, that are, uniformly and strongly continuous semigroups. Respective properties are presented as follows.

Theorem 3.5. *The linear operator A is the generator of a uniformly continuous semigroup if and only if A is a bounded operator. Thus the semigroup is expressed as $e^{tA} := \sum_{i=0}^{\infty} \frac{(At)^i}{i}$, $t \geq 0$.*

Corollary 3.6. *Assume that $S(t) = e^{tA}$ is a uniformly continuous semigroup with A as an infinitesimal generator . Then,*

$$\frac{d}{dt} S(t) = AS(t) = S(t)A$$

and

$$x(t) = S(t)x_0, \quad t > 0 \quad \textit{with } x(0) = x_0 \in X.$$

From Theorem 3.5, a unbounded generator A for a strong continuous semigroup is not uniformly continuous because the convergence of the series is not likely. Contrariwise, the existence of $e^{At} = \lim_{n \to \infty} \left(1 - \frac{tA}{n}\right)^{-n}$ under some assumptions on the operator A has made A be the generator of a C_0-semigroup. The following defintions leads us to the Hille-Yosida theorem.

Definition 3.7. A semigroup $\{S(t)\}$ is called a contraction semigroup if it satisfies

$$\|S(t)\| \leq 1, \forall t \geq 0.$$

Definition 3.8. Let A: $D(A) \rightarrow X$ be a linear operator

1. The resolvent set $\rho(A)$ of A is a set of complex numbers defined by

$$\rho(A) = \{\lambda \in \mathcal{C}, \quad \lambda I - A : D(A) \rightarrow X \quad \text{is invertible}\}.$$

2. For $\lambda \in \rho(A)$, the operator $R(\lambda, A)$ defined by

$$R(\lambda, A) := (\lambda I - A)^{-1}$$

is called the resolvent operator.

Theorem 3.9. *A linear unbounded operator A is the generator of a strongly continuous contraction semigroup if and only if*

1. *A is closed and densely defined,*
2. *For all $\lambda > 0$,*

$$\|R(\lambda, A)x\| \leq \frac{\|x\|}{\lambda}.$$

The Hille-Yosida theorem as stated in Theorem 3.9 plays a important role in characterizing the operator generating a C_0-semigroup. More informations, and proofs included, can be found in [34].

Indeed, the semigroup theory can effectively determine if a problem is well-posed. Hence the following theorem.

Theorem 3.10. *The IVP as defined by Equation (3.14) is well-posed if and only if A is the generator of a semigroup $S(t)$ and the unique solution given by $x(t) = S(t)x_0$, for $x_0 \in D(A)$ satisfying*

$$x \in C\left([0, \infty); D(A)\right) \cap C^1\left([0, \infty); X\right).$$

3.3 Analysis of Caputo fractional evolution equations

The concept of fractional derivative has been extended to LEE, which originated the notion of solution operators comparatively to semigroup for classical EE. Solution operators are introduced by Bazhlekova in [40] as a generalization of semigroup. This generalization implies that solution operators can also be whether uniformly or strongly continuous under certain conditions. Their representation plays a important role for numerical approximation of the solution. In this section, we study the fractional evolution equation with CFD by providing related defintions of solution operator and theorems to demonstrate well-posedness of extended LEE with CFD.

3.3.1 Solution operators

Consider the Cauchy problem for the fractional evolution equation of order α with $0 < \alpha < 1$,

$$\begin{aligned} {}^{CF}D_{0,t}^{\alpha} x(t) &= Ax(t), \quad t > 0 \\ x(0) &= x_0, \quad A \in X, \end{aligned}$$

where ${}^{CF}D_{0,t}^{\alpha}$ is the Caputo fractional derivative of order α, and A is a linear and bounded operator defined on a Banach space X.

The problem defined in Equation 3.15, is well-posed if and only if the Voltera integral equation

$$x(t) = x_0 + \int_0^t g_\alpha (t - \tau) \, Au(\tau) \, d\tau. \tag{3.15}$$

with $g_\alpha(t)$ is defined for $\alpha > 0$, by

$$g_\alpha = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)} & t > 0 \\ 0, & t \le 0 \end{cases} \tag{3.16}$$

is well posed.

Definition 3.11. A family $\{S_\alpha(t)\}_{t \ge 0}$ of bounded linear operators on X is called a uniformly continuous solution operator for Equation 3.15 on X, if the following conditions are satisfied

1. $S_\alpha(t)$ is a uniformly continuous functionfor $t \ge 0$ and $S_\alpha(0) = I$, where I is the identity operator on X.
2. $AS_\alpha(t)\, x_0 = S_\alpha(t)\, Ax_0$, $\forall x_0 \in X, t \ge 0$
3. $S_\alpha(t)\, x_0$ is a solution of Equation 3.15 $\forall x_0 \in X, t \ge 0$.

Definition 3.12. A linear operator A defined by

$$Ax = \Gamma(\alpha + 1) \lim_{t \to 0} \frac{S_\alpha(t)x - x}{t^\alpha}, \text{for all} \quad x \in X$$

is called infinitesimal generator of a uniformly continuous solution operator $S(t)_\alpha$,$\alpha > 0$ and $t \geq 0$ for Equation 3.15.

Another definition of the generator A is given by

$$Ax = \left({}^{CF}D^{\alpha}_{0,t}S_\alpha\right)(t)c|_{t=0},$$

since $J^{\alpha CF}_t D^{\alpha}_{0,t}S_\alpha(t)x = S_\alpha(t)x - x$ and for all functions $C\left([0,\infty);X\right)$ holds

$$\lim_{t\to 0}\frac{J^{\alpha}_t x(t)}{g_{\alpha+1}(t)} = x(0)$$

where J^{α}_t, $\alpha \geq 0$, is the fractional integral defiined for a function f by

$$J^{\alpha}_t f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau$$

with $J^0 = I$, I the indentity operator.

Definition 3.13. The solution operator $S_\alpha(t)$ is called exponentially bounded if there exists $\omega \geq 0$ and $M \geq 1$ such that

$$\|S_\alpha(t)\| \leq Me^{\omega t}, \quad t \geq 0.$$

Theorem 3.14. *Let $\alpha > 0$. Then the exponentially bounded uniformly continuous solution operator $S_\alpha(t)$ is the solution operator for the problem defined in Equation 3.15 if and only if A is a bounded operator in X.*

Moreover, every solution operator has a unique infinitesimal generator. Hence if the solution operator is according to Defintion 3.13, then its infinitesimal generator is a bounded linear operator given by

$$S_\alpha(t) = E_\alpha(t^\alpha A) = \sum_{k=0}^{\infty}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, \quad \alpha > 0, t \geq 0.$$

Corollary 3.15. *Assume that $S_\alpha(t) = E_\alpha(t^\alpha A)$ for $t \geq 0$ is a uniformly continuous solution operator with the operator A as an infinitesimal generator. Then,*

$${}^{CF}D^{\alpha}_{0,t}S_\alpha(t) = AS_\alpha(t) = S_\alpha(t)A$$

and

$$x(t) = S_\alpha(t)x_0, \quad t > 0 \quad \textit{with} \quad x(0) = x_0 \in X.$$

3.4 Summary

In this chapter, ordinary evolution equations and Caputo-fractional linear evolution equations were discussed in Section 3.2 and Section 3.3 respectively. The analysis of ordinary evolution equations was elaborated using the semigroup concept. This gave a better understanding of Semigroup theory and introduced their generalization under the form of solution operators for fractional differential equations. The well-posedness for these differential equations was also discussed.

Chapter 4

Application of the new Caputo-Fabrizio time fractional derivative to linear evolution equations

"We know very little, and yet it is astonishing that we know so much, and still more astonishing that so little knowledge can give us so much power"

Bertrand Russell

4.1 Introduction

After reviewing the literature that forms the base for the theory of fractional integrals and derivatives and also discussing the theory of solution operators as analysis tool to both ordinary and fractional evolution equations, we can now approach the fractional evolution equation with Caputo-Fabrizio fractional derivative (CFFD). We have reviewed in the previous chapter the fractional evolution equation with Caputo fractional derivative which was investigated by number of authors in the past, including Prüss [41] and Bazhlekova [7, 8]. In this chapter, the linear fractional evolution equation with non-singular kernel is formulated, making use of the CFFD. The notion of solution operator is shown to play a fundamental role in the study of such model. The issues of solution operator generated by our operator, the conditions under which this solution operator is analytic, and its representation in terms of the corresponding generator are analyzed.

4.2 Preliminaries

The goal of this chapter is to deeply investigate linear evolution systems using the CFFD, the newly introduced fractional derivative with non-singular kernel. Here the notation adopted is different from the one used in the Chapter 2 since in various applications x is used as a spatial variable and not as a function. Let us recall the definition of CFFD developed and proposed by Caputo and Fabrizio [14] defined as follows:

Definition 4.1 (**Caputo-Fabrizio derivative with fractional order (CFFD)**). Let u be a function in $H^1(0;+\infty)$; $\alpha \in [0;1]$ then, the new Caputo derivative of fractional order α is defined as:

$$ {}^{CF}D_t^{\alpha}u(x,t) = \frac{M(\alpha)}{(1-\alpha)} \int_0^t \dot{u}\,(x,\tau)\, \exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right) d\tau, \tag{4.1}$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. But, for the function that does not belong to $H^1(a;b)$, we defined its Caputo-Fabrizio fractional as

$$ {}^{CF}D_t^{\alpha}u(t) = \frac{\alpha M(\alpha)}{(1-\alpha)} \int_0^t (u(t) - u\,(\tau))\, \exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right) d\tau. \tag{4.2}$$

The definition of the CFFD was improved by Losada and Nieto [20] to become

$$ {}^{CF}D_t^{\alpha}u(t) = \frac{(2-\alpha)M(\alpha)}{2\,(1-\alpha)} \int_0^t \dot{u}\,(\tau)\, \exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right) d\tau. \tag{4.3}$$

Moreover, The anti-derivative associated to the CFD is defined as [11, 14, 20]

$$ {}^{CF}I_t^{\alpha}u(x,t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(x,t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t u\,(x,\xi)\, d\xi, \tag{4.4}$$

where $\alpha \in [0,1]$ $t \geq 0$.

Hence the extended fractional differential equation of the homogeneous LEE in (1.1) with CFFD as the derivative without kernel is then defined by

$$ \begin{aligned} &{}^{CF}D_t^{\alpha}u(x,t) = [Au(\cdot,t)](x), \quad 0 < \alpha \leq 1, \;\; t > 0 \\ &u(x,0) = u_0(x)\;, \;\; x > 0, \end{aligned} \tag{4.5}$$

where A is a certain differential and (or) integral expression, that can be evaluated at any point $x > 0$ for functions u belonging to a certain subset of the domain of A.

4.3 Solution operators and well-posedness for linear evolution models with non-singular kernel

Our analysis based on the solution operators is done in this section. We define a Banach space H endowed with the norm $\|\cdot\|_H$, express the model (4.5) in the simplified form

$$\begin{aligned} {}^{CF}D_t^{\alpha}u(t) &= Au(t), \quad 0<\alpha\leq 1, \;\; t>0 \\ u(0) &= u_0 \end{aligned} \tag{4.6}$$

and define the domain

$$D(A) := \{v \in H : Av \in H\} \tag{4.7}$$

on which the realization operator A is defined. The initial value problem (4.6) is called the abstract Cauchy problem associated to $(A, D(A))$ (with the linear operator $A : D(A) \subset H \longrightarrow H$) and the initial value u_0.

Definition 4.2 (**Strong solution for the initial value problem** (4.6))**.** A function in $C(\mathbb{R}_+, H)$; $\alpha \in [0; 1]$ is called a strong solution of the initial value problem (4.6) if $u \in C(\mathbb{R}_+, D(A))$, $\quad u \in C^1(\mathbb{R}_+, H)$, $\quad u(0) = u_0$ and ${}^{CF}D_t^{\alpha}u(t) = Au(t)$ holds on $\mathbb{R}_+$.

We note that the operator A can take various form,, making the initial value problem (4.6) specific to some kind of phenomena as we will see in Section 4.6.1.

Definition 4.3. The abstract Cauchy problem (4.6) is said to be well-posed if for every $x \in D(A)$, there exists a unique strong solution $u(x, t)$ of (4.6), the operator A has dense domain and for every sequence $(x_k)_{k\in\mathbb{N}} \subset D(A)$ satisfying $\lim_{k\to\infty} x_k = 0$, one has $\lim_{k\to\infty} u(t; x_k) = 0$ uniformly in compact intervals of type $[0, t_0]$..

Now we consider the problem (4.6). If we apply on both sides the anti-derivative (4.4), we get

$$u(x,t) - u(x,0) =^{CF} I_t^{\alpha} Au(x,t)$$

yielding

$$u(x,t) - u(x,0) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} Au(x,t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t Au\,(x,\xi)\,d\xi. \tag{4.8}$$

Equation (4.8) represents the Volterra integral equation corresponding to (4.6).

Remark 4.4. It is well-known [41, Definition 1.2] that the initial value problem (4.6) is well-posed if and only if the Volterra version (4.8) is well-posed.

Definition 4.5 (**Solution operators**)**.** Consider an operator A applying in the fractional model (4.6) and defined in the Banach space U. A family $\{T_\alpha(t)\}_{t>0}$ of bounded operators on U is called a solution operator of the initial value problem (4.6) if

1. $T_\alpha(0) = I_U$;

2. $T_\alpha(t)$ is strongly continuous for every $t \geq 0$;

3. $AT_\alpha(t)u = T_\alpha(t)Au$ for all $u \in D(A)$;

4. $T_\alpha(t)D(A) \subset D(A)$;

5. $T_\alpha(t)u$ is a solution of the model (4.6) for all $u \in D(A)$, $t \geq 0$.

Remark 4.6. As in [41], we say that the initial value problem (4.6) is well-posed if it admits a solution operator.

Definition 4.7. The solution operator $T_\alpha(t)$ is called exponentially bounded if there exist constants $\omega \geq 0$ and $M \geq 1$ such that

$$\|T_\alpha(t)\|_U \leq Me^{wt}, \quad t \geq 0. \tag{4.9}$$

Moreover, if the system (4.6) admits a solution operator $\{T_\alpha(t)\}_{t\geq 0}$ satisfying (4.9), then we say that the operator $A \in \mathcal{C}^\alpha(M, \omega)$.

And also $\{T_\alpha(t)\}_{t\geq 0}$ is said to be contractive if

$$\|T_\alpha(t)\|_U \leq 1, \tag{4.10}$$

and we say $A \in \mathcal{C}^\alpha(1, 0)$.

In concluding this section, we have highlighted the importance of solution operators with respect to well-posedness of Problem (4.6). Based on Remarks 4.4 and 4.6 we will demonstrate the existence of a solution operator for Problem (4.6) in the next section. And this will be done through the famous condition named Lipschitz condition.

4.4 Well-posedness via Lipschitz condition

In mathematical analysis, the theory of fixed points has played an important role in showing the existence and the uniqueness of the solutions of both integral and differential equations. Related theorems provides conditions under which generalized differential equations or mappings are solvable. Results obtained in fixed point theory are also achieved by using contraction mapping, which is a special type of Lipschiptzian maps on a metric space. In general sense, the map $f : X \to X$ is said to be a Lipschitzian if there exist a constant $L > 0$ such that for all x, y in X,

$$d\left(f(x), f(y)\right) \leq Ld\left(x, y\right), \tag{4.11}$$

where L is called the Lipschitz constant of the function f. Usually when the Lipschitz constant is less than one, the mapping described by f is then referred to as a contractive map. Hence the Lipschitz condition, originated from Rudolf Lipschitz's investigation in 1876, is defined precisely as follows:

Definition 4.8 (Lipschitz condition). The mapping between two metric spaces: $f : X- > Y$ is said to satisfy the **Lipschitz condition**, or be **Lipschitz continuous** or **L-Lipschitz** if there exists a real constant L such that

$$d_Y(f(x), f(y)) \leq L\, d_X(x, y), \tag{4.12}$$

for all $x, y \in X$. L is the Lipschitz constant.

Theorem 4.9 (Existence and Uniqueness). *Suppose that $D = \{(u(t), t) : a \leq t \leq b$ and $u(t) \in R\}$ and that $f(u(t), t)$ is continuous on D. If f satisfies a Lipschitz condition on D in the variable $u(t)$, then the IVP: $u^{'}(t) = f(u(t), t); a \leq t \leq b, \quad u(a) = \beta$, is well-posed.*

With reference to Theorem 4.9, one could argue why well-posedness since the third condition of well-posedness, which is the continuous dependence on the parameters, is not explicitly mentioned in the theorem. Indeed it is implicitly mentioned since if a perturbed problem is associated to the original problem in the theorem, then the perturbed problem still admit a unique solution. Consequently, though Theorem 4.9 is for both existence and uniqueness properties, it is as well for well-posedness as far it concerned initial value problems. Our analysis then tests for existence and uniqueness of a solution of Problem (4.6).

4.4.1 Analysis results

Let us now define the operator

$$\mathcal{O}(x, t, \epsilon_A, u) = Au(x, t). \tag{4.13}$$

where ϵ_A is any real parameter depending on the expression of A.

Proposition 4.10. *If the Lipschitz condition with respect to u holds for the operator $\mathcal{O}(x, t, \epsilon_A, u)$ defined in (4.13) then, the abstract Cauchy problem (4.6) admits a solution that is continuous. Furthermore, this solution is unique if the following condition holds*

$$1 - \frac{2L(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2Lt\alpha}{(2-\alpha)M(\alpha)} > 0$$

Proof. To prove the **existence of a continuous solution**, we consider (4.8):

$$u(x,t)-u(x,0) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\mathcal{O}(x,t,\epsilon_A,u)+\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \mathcal{O}(x,\tau,\epsilon_A,u)d\tau, \quad (4.14)$$

that suggests the following recurrence formula

$$\begin{aligned} u_0(x,t) &= u(x,0) \\ u_n(x,t) &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\mathcal{O}(x,t,\epsilon_A,u_{n-1}) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \mathcal{O}(x,\tau,\epsilon_A,u_{n-1})d\tau. \end{aligned} \quad (4.15)$$

Let

$$\bar{u}(x,t) = \lim_{n\to\infty} u_n(x,t). \quad (4.16)$$

We aim to show that $\bar{u}(x,t) = u(x,t)$ is a solution that is continuous. Let us set

$$G_n(x,t) = u_n(x,t) - u_{n-1}(x,t) \quad (4.17)$$

It is obvious that

$$u_n(x,t) = \sum_{m=0}^{n} G_m(x,t)$$

Furthermore, in a more detailed way we have

$$\begin{aligned} G_n(x,t) = {} & \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left[\mathcal{O}(x,t,\epsilon_A,u_{n-1}) - \mathcal{O}(x,t,\epsilon_A,u_{n-2})\right] \\ & + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \left[\mathcal{O}(x,\tau,\epsilon_A,u_{n-1}) - \mathcal{O}(x,\tau,\epsilon_A,u_{n-2})\right] d\tau. \end{aligned} \quad (4.18)$$

Taking the norm of the later equation gives

$$\begin{aligned} \|G_n(x,t)\| &= \|u_n(x,t) - u_{n-1}(x,t)\| \\ &\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\|\mathcal{O}(x,t,\epsilon_A,u_{n-1}) - \mathcal{O}(x,t,\epsilon_A,u_{n-2})\| \\ &\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)}\left\|\int_0^t (\mathcal{O}(x,\tau,\epsilon_A,u_{n-1}) - \mathcal{O}(x,\tau,\epsilon_A,u_{n-2}))d\tau\right\| \\ &\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\|\mathcal{O}(x,t,\epsilon_A,u_{n-1}) - \mathcal{O}(x,t,\epsilon_A,u_{n-2})\| \\ &\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \|\mathcal{O}(x,\tau,\epsilon_A,u_{n-1}) - \mathcal{O}(x,\tau,\epsilon_A,u_{n-2})\| d\tau \end{aligned} \quad (4.19)$$

Using the Lipschitz condition for $\mathcal{O}$ yields

$$\|G_n(x,t)\| \le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L\|u_{n-1} - u_{n-2}\| + \frac{2L\alpha}{(2-\alpha)M(\alpha)}\int_0^t \|u_{n-1} - u_{n-2}\|d\tau$$

equivalent to

$$\|G_n(x,t)\| \leq \frac{2(1-\alpha)L}{(2-\alpha)M(\alpha)}\|G_{n-1}\| + \frac{2L\alpha}{(2-\alpha)M(\alpha)}\int_0^t \|G_{n-1}\|d\tau \tag{4.20}$$

The recursive's principle from (4.20) gives

$$\|G_n(x,t)\| \leq \left[\left(\frac{2(1-\alpha)L}{(2-\alpha)M(\alpha)}\right)^n + \left(\frac{2L\alpha t}{(2-\alpha)M(\alpha)}\right)^n\right] u(x,0),$$

which proves that the solution exists and is continuous. To show that

$$u(x,t) = \lim_{n\to\infty} u_n(x,t)$$

is the solution of the model (4.6), we let

$$Z_n(x,t) = \bar{u}(x,t) - u_n(x,t) \qquad \text{for} \qquad n \in \mathbb{N}.$$

Hence, from (4.16), the difference $Z_n(x,t)$ between $\bar{u}(x,t)$ and $u_n(x,t)$ should tend to zero as $n \to \infty$. Indeed

$$\begin{aligned}\bar{u} - u_{n-1} = {} & \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left[\mathcal{O}(x,t,\epsilon_A,u) - \mathcal{O}(x,t,\epsilon_A,u_n)\right] \\ & + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \left[\mathcal{O}(x,\tau,\epsilon_A,u) - \mathcal{O}(x,\tau,\epsilon_A,u_n)\right]d\tau,\end{aligned} \tag{4.21}$$

giving

$$\begin{aligned}\|\bar{u}(x,t) - u_{n+1}\| \leq {} & \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\|\mathcal{O}(x,t,\epsilon_A,u) - \mathcal{O}(x,t,\epsilon_A,u_n)\| \\ & + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \|\mathcal{O}(x,\tau,\epsilon_A,u) - \mathcal{O}(x,\tau,\epsilon_A,u_n)\|d\tau \\ \leq {} & \frac{2L(1-\alpha)}{(2-\alpha)M(\alpha)}\|u - u_n\| + \frac{2Lt\alpha}{(2-\alpha)M(\alpha)}\int_0^t \|u - u_n\|d\tau \\ \leq {} & \frac{2L(1-\alpha)}{(2-\alpha)M(\alpha)}\|Z_n\| + \frac{2Lt\alpha}{(2-\alpha)M(\alpha)}\int_0^t \|Z_n\|d\tau.\end{aligned} \tag{4.22}$$

Then indeed when $n \to \infty$, then $Z_n \to 0$ and the right hand side gives

$$\lim_{n\to\infty} u_n = \bar{u}.$$

We can take $u(x,t) = \bar{u}(x,t)$ as a solution of (4.6) that is continuous. Furthermore, applying the lipschitz condition for $\mathcal{O}$, we have the following:

$$\begin{aligned} u(x,t) - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\mathcal{O}(x,t,\epsilon_A,u) - \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \mathcal{O}(x,t,\epsilon_A,u)d\tau \\ = R_n(x,t) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left[\mathcal{O}(x,\tau,\varepsilon_A,u_{n-1}) - \mathcal{O}(x,t,\epsilon_A,u)\right] \\ + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \left[\mathcal{O}(x,\tau,\epsilon_A,u_{n-1}) - \mathcal{O}(x,t,\epsilon_A,u)\right]d\tau. \end{aligned} \tag{4.23}$$

This yields

$$\begin{aligned} \left\| u(x,t) - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\mathcal{O}(x,t,\epsilon_A,u) - \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \mathcal{O}(x,t,\epsilon_A,u)d\tau \right\| \\ = \|G_n(x,t)\| + \left[\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\theta t\alpha}{(2-\alpha)M(\alpha)}\right]\|G_{n-1}(x,t)\|. \end{aligned} \tag{4.24}$$

Passing to the limit when $n \to 0$ and considering the initial condition, we have

$$u(x,t) = u(x,0) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\mathcal{O}(x,t,\epsilon_A,u) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \mathcal{O}(x,t,\epsilon_A,u)d\tau,$$

proving the existence of a continuous solution under Proposition 4.10.

To prove the **uniqueness of the continuous solution**, we consider u and v to be two different solutions of the model (4.6) then, we aim to show that

$$u = v.$$

The lipschitz condition for $\mathcal{O}$ yields

$$\|u-v\| \leq \frac{2L(1-\alpha)}{(2-\alpha)M(\alpha)}\|u-v\| + \frac{2Lt\alpha}{(2-\alpha)M(\alpha)}\|u-v\|, \tag{4.25}$$

rearranged to be

$$\|u-v\|\left(1 - \frac{2L(1-\alpha)}{(2-\alpha)M(\alpha)} - \frac{2Lt\alpha}{(2-\alpha)M(\alpha)}\right) \leq 0.$$

With the assumption

$$1 - \frac{2L(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2Lt\alpha}{(2-\alpha)M(\alpha)} > 0$$

then, $\|u-v\| = 0$ meaning

$$u = v$$

and the uniqueness of the solution is proved and so is Proposition 4.10. □

4.5 Well-posedness via the Picard $\mathcal{T}$-stability

From the previous section, we discussed the well-posedness of the Caputo-Fabrizio fractional evolution equation with respect to the Lipschitz approach. The properties such as existence and uniqueness were explicitly discussed but the continuous dependence on parameters not. In this section we approach the well-posedness property with the Picard $\mathcal{T}$-stability, which is very apparent in the functional analysis of metric spaces.

The notion of metric spaces, in which the word metric means measure, was first introduced by the French mathematician Maurice Frechet in 1906. Since then various theories were developed which include the metric (Banach) fixed point theory where successive approximations is used to prove both the existence and uniqueness of solutions to differential equations under appropriate conditions. The fixed point theorem also known as the Banach contraction principle is linked to the work of Picard. This theorem, from a numerical viewpoint, provides an error estimate for the Picard's iteration while approximating the fixed point, and solid information on the data dependence of that fixed point.

Definition 4.11 (Fixed point). The space X is said to have a fixed point property if for any continuous mapping $\mathcal{T} : X \to X$, there exists a point x in X such that $\mathcal{T}(x) = x$, which is stays invariant under the mapping $\mathcal{T}$.

In numerical form, the Picard iterative method $\{x_n\}$ is defined by

$$x_{n+1} = \mathcal{T} x_n, \quad n = 0, 1, 2, \dots, \tag{4.26}$$

is called Picard iterative method. It is also important to point out that when the mapping satisfies Lipschitz conditions then the sequence as defined in (4.26) converges to a unique fixed point. This has been proven in Section 4.4.

The stability of the Picard iterative method is guaranteed when the actual sequence and the approximate sequence converge both to a same fixed point. This implies that a small change in the data involved in the calculation process, produces a small change in the calculated value of the fixed point. The $\mathcal{T}$-stability of Picard iteration is applied here to demonstrate the data-dependence of the unique solution. This approach was used in [42] though the stability was named after the operator $\mathcal{K}$.

The concept of $\mathcal{T}$-stability was given by Harder and Hicks [43] as follows

Definition 4.12 ($\mathcal{T}$-stability). Let $(X, \|\|)$ be a Banach space, consider $\mathcal{T}$ a self-map of X and assume that $x_{n+1} = f(\mathcal{T}, x_n)$ define some iterative scheme involving the mapping $\mathcal{T}$. Suppose that the sequence $\{x_n\}$ converges to a fixed point q of $\mathcal{T}$. Let $\{y_n\}$

be an arbitrary sequence in X and define $d_n = \|y_{n+1} - f(\mathcal{T}, y_n)\|$ for $= 0, 1, 2, \ldots$. If $\lim\limits_{n\longrightarrow\infty} d_n = 0$ implies that $\lim\limits_{n\longrightarrow\infty} y_n = q$. Then the iterative scheme $x_{n+1} = f(\mathcal{T}, x_n)$ is said to be $\mathcal{T}$-stable.

Moreover, with the assumption that $\{y_n\}$ has a upper boundary and if all the conditions above hold for $x_{n+1} = \mathcal{T}x_n$, then it is called Picard's iteration and is $\mathcal{T}$-stable.

Lemma 4.13. *(see [44, 45]) Let $(X, \|\|)$ be a Banach space and $\mathcal{T}$ a self-map of X satisfying*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq C\|x - \mathcal{T}x\| + \gamma\|x - y\|$$

for all $x, y \in X$, where $0 \leq C,\ 0 \leq \gamma \leq 1$. With the assumption that $\mathcal{T}$ has a fixed point q. Then, $\mathcal{T}$ is Picard $\mathcal{T}$-stable.

4.5.1 Analysis results

Proposition 4.14. *Consider the self-map $\mathcal{T}$ defined by the recurrence relation*

$$\begin{aligned} \mathcal{T}(u_n(x,t)) &= u_{n+1}(x,t) \\ &= u_n(x,t) + {}^{CF}I_t^{\alpha}[\mathcal{O}(x,t,\epsilon_A,u)], \end{aligned} \tag{4.27}$$

then, it is $\mathcal{T}$-stable in $L^2(a,b)$

Proof. We start by showing that $\mathcal{T}$ has a fixed-point. Let $i, j \in \mathbb{N}$ then,

$$\|\mathcal{T}u_i(x,t) - \mathcal{T}u_j(x,t)\| = \|u_{i+1}(x,t) - u_{j+1}(x,t)\|$$

$$= \left\|u_i(x,t) + {}^{CF}I_t^{\alpha}[\mathcal{O}(x,t,\epsilon_A,u_i)] - u_j(x,t) - {}^{CF}I_t^{\alpha}[\mathcal{O}(x,t,\epsilon_A,u_j)]\right\|$$

where

$${}^{CF}I_t^{\alpha}u(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t u\,(\xi)\,d\xi,$$

is the fractional integral associated to the Caputo-Fabrizio fractional derivative defined in (4.4). Then,

$$\|\mathcal{T}u_i(x,t)-\mathcal{T}u_j(x,t)\| \leq \|u_i(x,t)-u_j(x,t)\|+\|{}^{CF}I_t^{\alpha}[\mathcal{O}(x,t,\epsilon_A,u_i)]-{}^{CF}I_t^{\alpha}[\mathcal{O}(x,t,\epsilon_A,u_j)]\|$$

Using the Lipschitz condition for the differential operator $\mathcal{O}$, there is a positive constant δ such that

$$\|\mathcal{T}u_i(x,t) - \mathcal{T}u_j(x,t)\| \leq \delta\|u_i(x,t) - u_j(x,t)\|,$$

and this proves that the Lipschitz condition holds for the nonlinear operator $\mathcal{T}$ and hence, it has a fixed point.
Lastly, if we take $C = 0$ and $\gamma = \delta$ then, the conditions of Lemma 4.13 holds for $\mathcal{T}$ which is therefore Picard $\mathcal{T}$-stable and the proof is complete. □

Remark 4.15. The analysis results also show the existence of a unique solution for the model (4.6) obtained via the fixed-point of $\mathcal{T}$ in the iterative scheme (4.21).

4.6 Validation of the fractional evolution equation with CFFD

In this section, we address the solvability of the extended fractional evolution equation with non singular kernel in kinetic, heat diffusion and shallow water wave applications.

4.6.1 Kinetic application

Different kinetic models expressed in terms of the new derivative of fractional order without any singular kernel is explored here. We however shall recall that the Laplace transform of the Caputo-Fabrizio fractional derivative is given by [14, 20, 46]

$$\mathcal{L}\left\{ {}^{CF}D_t^{\alpha}u(t), s\right\} = \frac{s\tilde{u}(x,s) - u_0(x)}{s + \alpha(1-s)} \tag{4.28}$$

where $\tilde{u}(x,s)$ is the Laplace transform $\mathcal{L}\left\{(u(x,t), s\right\}$ of $u(x,t)$.

1. The first type of kinetic model is the stationary one taking the form

$$\begin{cases} {}^{CF}D_t^{\alpha}u(t) = 0 \\ u(0) = u_0, \end{cases} \tag{4.29}$$

   Applying the Laplace transform (4.28) on both side of (4.29) yields

$$\mathcal{L}\left\{ {}^{CF}D_t^{\alpha}u(t), s\right\} = 0$$

   or

$$\frac{s\tilde{u}(s) - u_0}{s + \alpha(1-s)} = 0.$$

   Hence, $\tilde{u}(s) = \frac{u_0}{s}$. Applying the inverse Laplace transform $\mathcal{L}^{-1}\left\{\tilde{u}(s), t\right\}_{t>0} = u(t)$ leads to

$$u(t) = u_0 = \text{constant} \tag{4.30}$$

Thus, the new CFFD without singular kernel applied to (4.29) gives the well known traditional result of a constant. See Fig. 4.1 for graphical illustration.

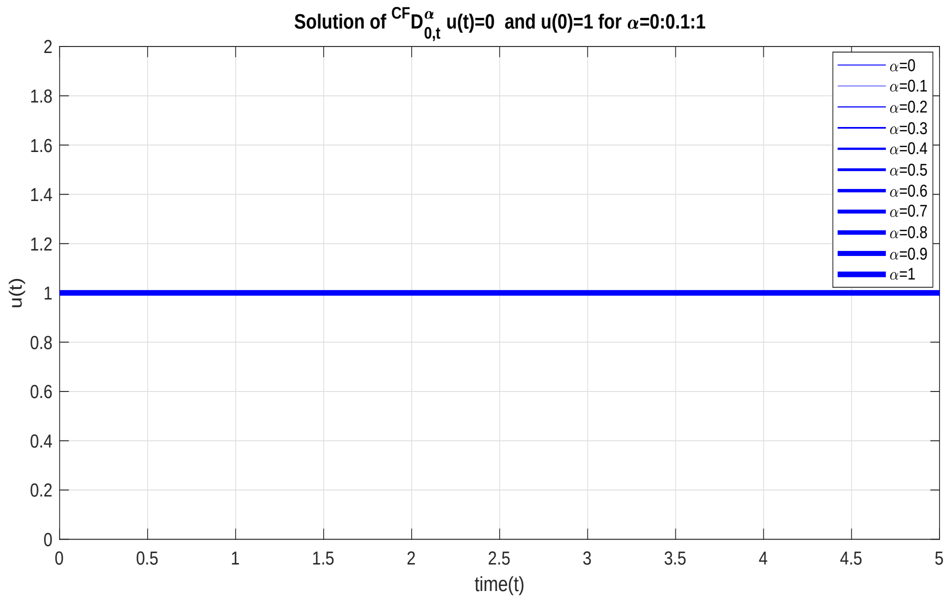

FIGURE 4.1: Solution of (4.29) for $u_0 = 1$ and arbitrary chosen α.

2. Another type of stationary model is given by

$$\begin{cases} {}^{CF}D_t^{\alpha} u(t) = K \\ u(0) = u_0, \end{cases} \tag{4.31}$$

Laplace transform (4.28) yields

$$\frac{(2-\alpha)M(\alpha)}{2}\,\frac{s\tilde{u}(s) - u_0}{s + \alpha(1-s)} = K\mathcal{L}\left\{1, s\right\}.$$

Developing and rearranging finally yields

$$\tilde{u}(s) = \frac{u_0}{s} + \frac{2(1-\alpha)K}{(2-\alpha)M(\alpha)}\mathcal{L}\left\{1, s\right\} + \frac{2K\alpha}{(2-\alpha)M(\alpha)}\,\frac{\mathcal{L}\left\{1, s\right\}}{s},$$

equivalent to

$$\tilde{u}(s) = \frac{u_0}{s} + \frac{2(1-\alpha)K}{(2-\alpha)M(\alpha)}\mathcal{L}\left\{1, s\right\} + \frac{2K\alpha}{(2-\alpha)M(\alpha)}\mathcal{L}\left\{\int_0^t d\xi, s\right\}.$$

The inverse Laplace transform applied the last equation gives the solution of (4.31):

$$u(t) = u_0 + \frac{2(1-\alpha)K}{(2-\alpha)M(\alpha)} + \frac{2\alpha K t}{(2-\alpha)M(\alpha)} \tag{4.32}$$

Remark 4.16. Note that the new derivatives were improved by imposing [20] that

$$\frac{2\,(1-\alpha)}{(2-\alpha)M(\alpha)}+\frac{2\alpha}{(2-\alpha)M(\alpha)}=1$$

Leading to

$$M(\alpha)=\frac{2}{(2-\alpha)}, \qquad 0<\alpha\leq 1.$$

Hence,

i. for $\alpha=1$, we have $M(\alpha)=2$ and we recover from solution (4.32) the traditional well known result

$$u(t)=u_0+Kt$$

ii. for $K=0$, we also recover from (4.32) the solution (4.30) of the first kinetic model (4.29).

Thus the graphical illustration of the stationary kinetic equation modeled with CFFD when $K=1$ is shown in Fig. 4.2.

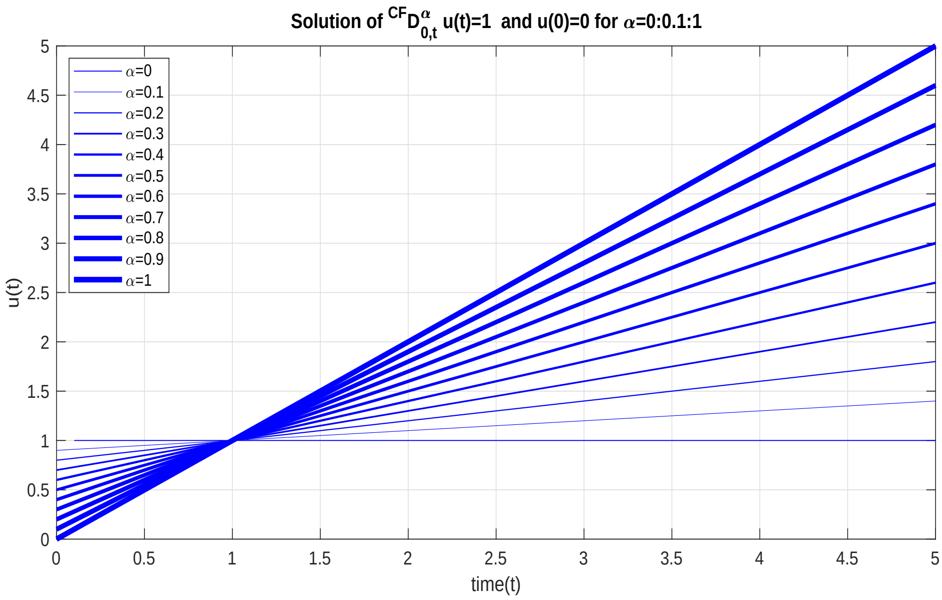

FIGURE 4.2: Solution of (4.31) for $u_0=0$, $K=1$ and arbitrary chosen α.

3. The third model to analyse is the fractional relaxation initial value problem

$$\begin{cases} {}^{CF}D_t^{\alpha}u(t)=-Ku(t) \\ u(0)=u_0, \end{cases} \tag{4.33}$$

Following the same steps as above gives

$$\frac{(2-\alpha)M(\alpha)}{2}\,\frac{s\tilde{u}(s)-u_0}{s+\alpha(1-s)}=-K\mathcal{L}\left\{u(t),s\right\}, \tag{4.34}$$

which solving for $\tilde{u}(s)$ gives

$$\tilde{u}(s) = \frac{(2-\alpha)M(\alpha)u_0}{s(2-\alpha)M(\alpha)+2K(s+\alpha(1-s))}. \tag{4.35}$$

Applying the inverse Laplace transform we obtain

$$u(t) = Au_0\frac{1}{C}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{B}{C}}, t\right\},$$

where

$$A = (2-\alpha)M(\alpha), \qquad B = 2K\alpha, \qquad C = (2-\alpha)M(\alpha)+2K(1-\alpha).$$

Thus, making use of the definition of inverse Laplace transform of standards functions, we obtain the solution of the fractional relaxation initial value problem (4.33):

$$u(t) = \chi_\alpha(t) = \frac{(2-\alpha)\,M(\alpha)u_0}{(2-\alpha)M(\alpha)+2K(1-\alpha)}\exp\left\{-\frac{2K\alpha t}{(2-\alpha)M(\alpha)+2K(1-\alpha)}\right\}. \tag{4.36}$$

Proposition 4.17. *Let K be any constant number and $0 < \alpha \leq 1$. The unique solution of the equation*

$$(1-Ka_\alpha)u + Kb_\alpha\int_0^t u(\xi)d\xi = (1+Ka_\alpha)u(0), \tag{4.37}$$

where $a_\alpha = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}$ and $b_\alpha = \frac{2\alpha}{(2-\alpha)M(\alpha)}$ is given by (4.36)

Proof. Another development of (4.34) using the properties of Laplace transform yields

$$u(t) = u_0 - \frac{2\,(1-\alpha)}{(2-\alpha)M(\alpha)}Ku(t) - \frac{2\alpha K}{(2-\alpha)M(\alpha)}\int_0^t u(\xi)d\xi,$$

which after development, is exactly (4.37). □

Remark 4.18. Two observations are made:

i. For $\alpha = 1$, we have $M(\alpha) = 2$ and we recover from solution (4.36) the traditional well known result

$$u(t) = u_0 e^{-Kt}.$$

ii. For $K = 0$, we also recover from (4.36) the solution (4.30) of the first kinetic model (4.29).

Thus the graphical illustration of the relaxation kinetic equation modeled with CFFD when $K = -1$ is shown in Fig. 4.3.

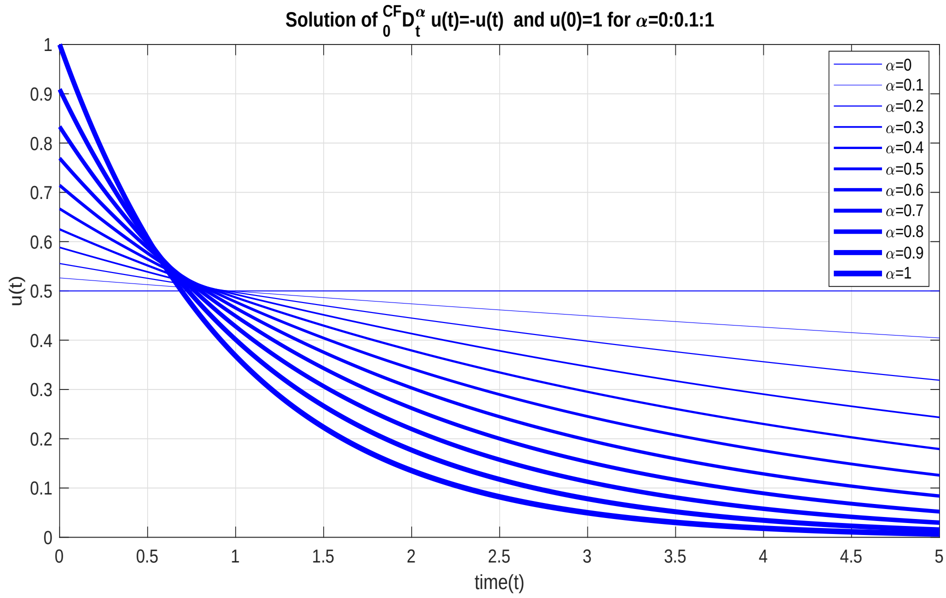

FIGURE 4.3: Solution of (4.33) for $u_0 = 1$, $K = -1$ and arbitrary chosen α.

4.6.2 Linear diffusion application

Let us consider the linear diffusion (heat) equation with time fractional Caputo-Fabrizio derivative of the form

$$ {}^{CF}D_t^{\alpha} u(x,t) = k\nabla^2 u(x,t), \quad 0<\alpha<1, \quad 0<t<T, \quad a \le x \le b, \tag{4.38}$$

under the initial condition

$$u(x,0) = h(x).$$

Applying the Laplace transformon both sides of (4.38) and taking into account the Laplace transform of the CFFD in (2.27) defined by

$$\mathcal{L}\{{}^{CF}D_{0,t}^{\alpha} u(x,t), s\} = \frac{s\mathcal{L}\{u(x,t),s\} - u(x,0)}{s+\alpha(1-s)}, \quad s>0, \tag{4.39}$$

we get

$$\frac{s\mathcal{L}\{u(x,t),s\} - u(x,0)}{s+\alpha(1-s)} = k\mathcal{L}\left\{\frac{\partial^2 u(x,t)}{\partial x^2}, s\right\} \tag{4.40}$$

or

$$\mathcal{L}\{u(x,t),s\} = \frac{u(x,0)}{s} + \frac{k(s+\alpha(1-s))}{s}\mathcal{L}\left\{\frac{\partial^2 u(x,t)}{\partial x^2}, s\right\}.$$

Using the inverse Laplace transform, we get

$$\begin{aligned} u(x,t) &= u(x,0) + \mathcal{L}^{-1}\{\frac{k(s+\alpha(1-s))}{s}\mathcal{L}\{\frac{\partial^2 u(x,t)}{\partial x^2}, s\}\} \\ &= u(x,0) + k\mathcal{L}^{-1}\{(1-\alpha-\frac{\alpha}{s})\mathcal{L}\{\frac{\partial^2 u(x,t)}{\partial x^2}, s\}\} \end{aligned}$$

which after simplification becomes

$$u(x,t) = u(x,0) + k(1-\alpha)\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right) - k\alpha \int_0^t \left(\frac{\partial^2 u(x,\tau)}{\partial x^2}\right) d\tau$$

or in shorthand notation as

$$u(x,t) = u(x,0) + k(1-\alpha)\nabla^2 u(x,t) - k\alpha \int_0^t \nabla^2 u(x,\tau) d\tau. \tag{4.41}$$

This equation (4.41) is comparable to (4.8) when considering the Losada and Nieto reformulated definition of the Caputo-Fabrizio fractional derivative when $(2-\alpha)M(\alpha) = 2$ for $0 < \alpha < 1$, and also $\mathcal{O} = k\nabla^2$. Consequently the following theorem holds for the considered linear diffusion model.

Theorem 4.19. *If the Lipschitz condition with respect to u holds for the operator $k\nabla^2$ then, the abstract Cauchy problem (4.38) admits a solution that is continuous. Furthermore, this solution is unique if the following condition holds*

$$1 - kL(1-\alpha) + kLt\alpha > 0.$$

Proof. As the model (4.38) is an example of the generalized fractional linear evolution equation defined in (4.6), and Proposition 4.10 is proven for the generalized fractional linear evolution equation in Proof 4.4.1, then Proof 4.4.1 also prove Theorem 4.19 when $\mathcal{O} = k\nabla^2$ and $(2-\alpha)M(\alpha) = 2$ for $0 < \alpha < 1$. □

Furthermore, a numerical scheme is derived to approximate the solution of the model (4.38). This is done by considering the iterative formal led by Equation 4.41 as follows.

$$\begin{aligned} &u(x,0) = u_0 \\ &u_{n+1}(x,t) = u_0 + k(1-\alpha)\nabla^2 u_n(x,t) - k\alpha \int_0^t \nabla^2 u_n(x,\tau) d\tau. \end{aligned} \tag{4.42}$$

Therefore the solution of 4.41 is as follows

$$u(x,t) = \lim_{n\to\infty} u_n(x,t).$$

In view of Equation (4.42) and based to the variational iteration method, the following correction function is constructed

$$u_{n+1}(x,t) = u_n(x,0) + k(1-\alpha)\nabla^2 u_n(x,t) - k\alpha \int_0^t \nabla^2 u_n(x,\tau)d\tau. \tag{4.43}$$

In the sequel, the linear mapping $\mathcal{T}$ defined by

$$\begin{aligned} \mathcal{T}(u_n(x,t)) &= u_{n+1}(x,t) \\ &= u_n(x,0) + k(1-\alpha)\nabla^2 u_n(x,t) - k\alpha \int_0^t \nabla^2 u_n(x,\tau)d\tau, \end{aligned} \tag{4.44}$$

is $\mathcal{T}$-stable in $L^2(a,b)$.This $\mathcal{T}$-stability is shown by following Proof 4.5.1 with respect to the considered linear operator $k\nabla^2$ and $(2-\alpha)M(\alpha) = 2$ for $0 < \alpha < 1$. Thus we show, by Lipschitz condition, that the linear map $\mathcal{T}$ has a fixed point, for $m, n \in \mathbb{N}$. Then we have

$$\begin{aligned} \|\mathcal{T}u_n(x,t) - \mathcal{T}u_m(x,t)\| = \big\|u_n - u_m + k(1-\alpha)\nabla^2 \{u_n - u_m\} \\ - k\alpha \int_0^t \nabla^2 \{u_n - u_m\} d\tau\|. \end{aligned}$$

Thus

$$\begin{aligned} \|\mathcal{T}u_n(x,t) - \mathcal{T}u_m(x,t)\| &\leq \|u_n - u_m\| + k(1-\alpha)\left\|\nabla^2 \{u_n - u_m\}\right\| \\ &\quad - k\alpha \left\|\int_0^t \nabla^2 \{u_n - u_m\} d\tau\right\| \\ &\leq \|u_n - u_m\| + k(1-\alpha)\beta_1\beta_2 \|u_n - u_m\| \\ &\quad - k\alpha\beta_1\beta_2 T\|u_n - u_m\| \\ &\leq (1 + k(1-\alpha - k\alpha T)\beta_1\beta_2) \|u_n - u_m\|. \end{aligned}$$

Therefore

$$\|\mathcal{T}u_n(x,t) - \mathcal{T}u_m(x,t)\| \leq L \|u_n(x,t) - u_m(x,t)\|, \tag{4.45}$$

where $L = 1 + k(1-\alpha - k\alpha T)\beta_1\beta_2$, proving that the Lipschitz condition holds for the linear operator $\mathcal{T}$ and implying that $\mathcal{T}$ has a fixed point.

Since $\mathcal{T}$ has a fixed point, then the following theorem confirms the well-posedness of the considered diffusion equation by providing a solution of (4.44).

Theorem 4.20. *Considering the iteration formula*

$$\begin{aligned} \mathcal{T}(u_n(x,t)) &= u_{n+1}(x,t) \\ &= u_n(x,0) + k(1-\alpha)\nabla^2 u_n(x,t) - k\alpha \int_0^t \nabla^2 u_n(x,\tau)d\tau. \end{aligned}$$

The linear map $\mathcal{T}$ is Picard $\mathcal{T}$-stable with respect to the norm of $L^2(a,b)$ if

$$1 + k(1 - \alpha - k\alpha T)\beta_1\beta_2 < 1 \tag{4.46}$$

Proof. Putting

$$C = 0 \quad \text{and} \quad \gamma = 1 + k(1 - \alpha - k\alpha T)\beta_1\beta_2,$$

shows that all conditions of Lemma 4.13 are satisfied. Hence the linear map is Picard $\mathcal{T}$-stable. □

An illustrative example is presented here for the heat problem given by

$$\begin{cases} {}^{CF}D_t^{\alpha}u(t) = \frac{\partial^2 u(x,t)}{\partial x^2}), \quad 0 < \alpha < 1, \quad 0 \le t \le 0.5, \quad 0 \le x \le 2, \\ u(x,0) = sin(x), \end{cases} , \tag{4.47}$$

whose the exact ordinary solution when $\alpha = 1$ is

$$u(x,t) = e^{-t^2} sin(x)$$

shown in Fig. 4.4.

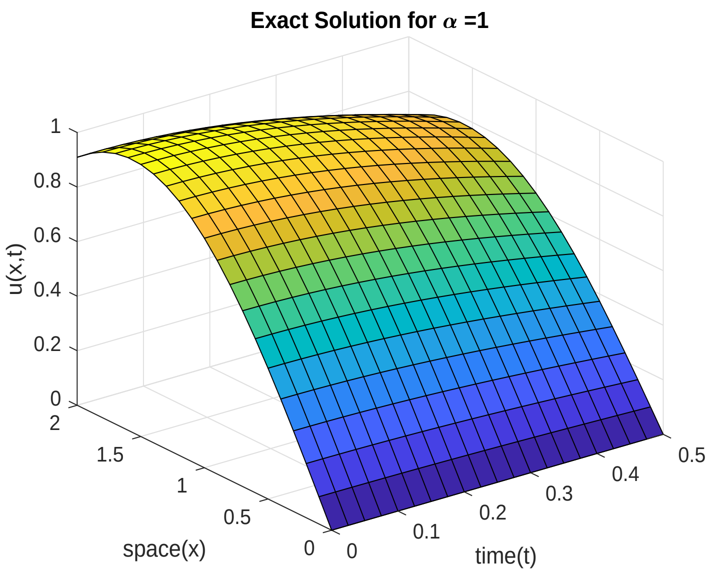

FIGURE 4.4: Exact solution with ordinary order ($\alpha = 1$).

An approximate solution of (4.6.2) for $\alpha = 0.1$ is shown in Fig. 4.5 for $n = 1, 5, 10$. Following the iterative scheme in 4.42, the solution $u(x,t)$ is complete for n sufficiently large ($n \to \infty$). By observing closely Fig. 4.5, the amplitudes of $u_1(x,t)$, though increasing, reduces with respect to time when comparing them with the ones of $u_{10}(x,t)$.

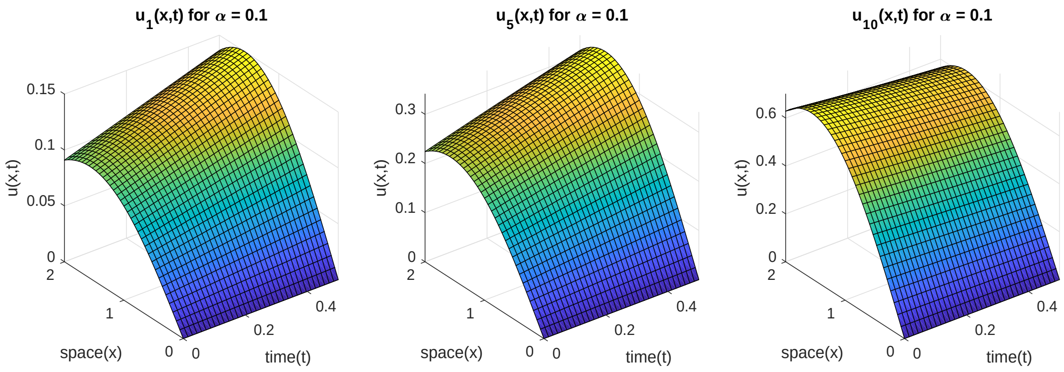

FIGURE 4.5: Approximate solution with fractional order ($\alpha = 0.1$) for $n = 1, 5, 10$.

The same trend as in Fig. 4.5 is observed in Fig. 4.6 and Fig. 4.7 but at a very slow paste.

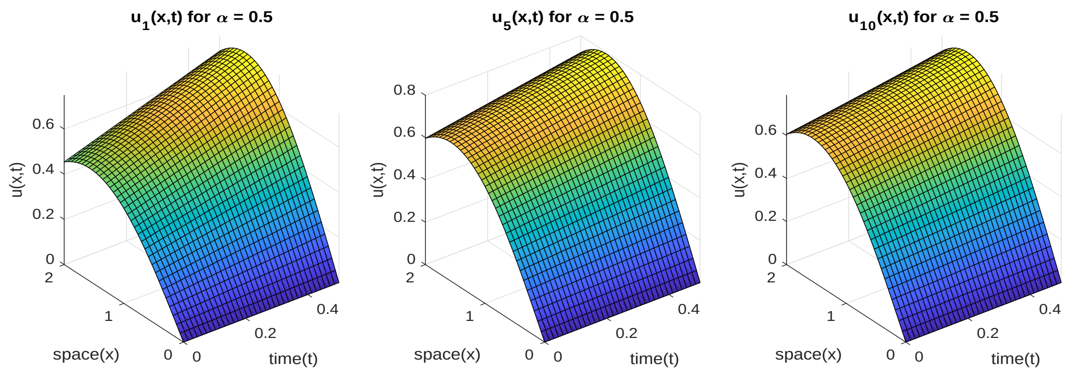

FIGURE 4.6: Approximate solution with fractional order ($\alpha = 0.5$) for $n = 1, 5, 10$.

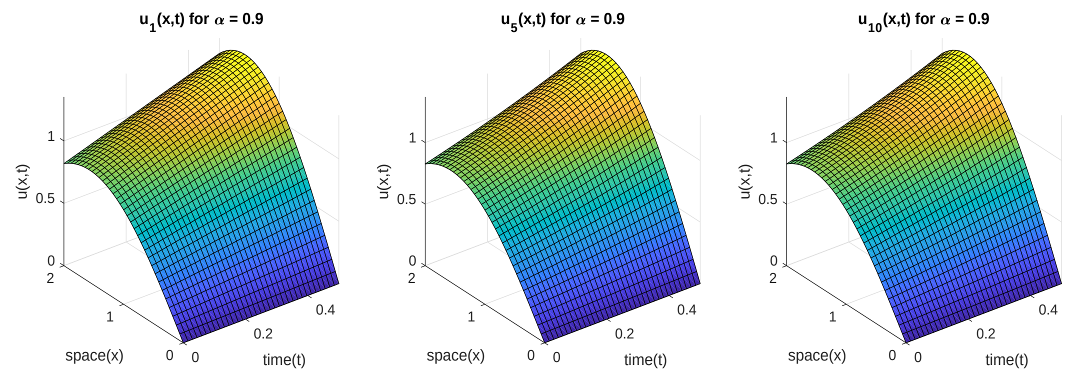

FIGURE 4.7: Approximate solution with fractional order ($\alpha = 0.9$) for $n = 1, 5, 10$.

Moreover, we propose to use another method named Laplace Homotopy Analysis method (LHAM) to solve the CFFD-based linear diffusion equation. This method was proven to be effective in finding approximate analytical solutions of fractional differential equations. These solutions are accurate and fast convergent obtained in few iterations.

Applying once again the Laplace transform on both sides of (4.38), taking into account the Laplace transform of the CFFD in (2.27), and solving for Laplace transform of $u(x,t)$, we get

$$\mathcal{L}\{u(x,t),s\} = \frac{u(x,0)}{s} + \frac{k(s+\alpha(1-s))}{s}\mathcal{L}\left\{\frac{\partial^2 u(x,t)}{\partial x^2}, s\right\}, \tag{4.48}$$

whose the corresponding homotopy is constructed as follows:

$$\Phi(x,s) = \frac{u(x,0)}{s} + p\frac{k(s+\alpha(1-s))}{s}\left(\frac{\partial^2}{\partial x^2}\right)\Phi(x,s), \tag{4.49}$$

where $\Phi(x,s) = \mathcal{L}\{u(x,t),s\}$.

The solution of (4.6.2) is obtained based on the hypothesis that it is expressed as

$$\Phi(x,s) = \sum_{i=0}^{\infty} p^i \Phi_i(x,s). \tag{4.50}$$

By substituting (4.6.2) into (4.48), we obtain

$$\sum_{i=0}^{\infty} p^i \Phi_i(x,s) = \frac{u(x,0)}{s} + p\frac{k(s+\alpha(1-s))}{s^{n+1}}\left(\frac{\partial^2}{\partial x^2}\right)\sum_{i=0}^{\infty} p^i \Phi_i(x,s), \tag{4.51}$$

which by identification of coefficients of powers of p in both sides of (4.51) yields to the following:

$$\begin{aligned}
&p^0: \quad \Phi_0(x,s) = \frac{u(x,0)}{s}\\
&p^1: \quad \Phi_1(x,s) = \frac{k(s+\alpha(1-s))}{s}\left(\frac{\partial^2}{\partial x^2}\right)\Phi_0(x,s)\\
&p^2: \quad \Phi_2(x,s) = \frac{k(s+\alpha(1-s))}{s}\left(\frac{\partial^2}{\partial x^2}\right)\Phi_1(x,s)\\
&p^3: \quad \Phi_3(x,s) = \frac{k(s+\alpha(1-s))}{s}\left(\frac{\partial^2}{\partial x^2}\right)\Phi_2(x,s)\\
&\vdots\\
&p^i: \Phi_i(x,s) = \frac{k(s+\alpha(1-s))}{s}\left(\frac{\partial^2}{\partial x^2}\right)\Phi_{i-1}(x,s)\\
&p^{i+1}: \Phi_{i+1}(x,s) = \frac{k(s+\alpha(1-s))}{s}\left(\frac{\partial^2}{\partial x^2}\right)\Phi_i(x,s).
\end{aligned}$$

Thus as p tends to 1, $\Phi_{i+1}(x,s)$ is the approximate solution of (4.6.2) yielding the solution of (4.38) in the s-domain given by

$$U_n(x,s) = \sum_{i=0}^{n} \Phi_i(x,s). \tag{4.52}$$

Hence, the inverse Laplace transform of (4.52) gives the approximate solution of (4.38),

$$u_n(x,t) = \mathcal{L}^{-1}\left\{U_n(x,s)\right\}. \tag{4.53}$$

So the accuracy of the approximate solution $(u_n(x,t))$ with respect the exact analytical solution $(u_{exact}(x,t))$ can be calculated by relative error, $re\,(\%)$ as follows:

$$re\,(\%) = |\frac{u_n(x,t) - u_{exact}(x,t)}{u_{exact}(x,t)}| \text{ x } 100. \tag{4.54}$$

An Illustrative example of application of LHAM to the linear diffusion equation as defined in (4.6.2) is presented here.

Using the LHAM, we have:

$$\begin{aligned}
&p^0: \quad \Phi_0(x,s) = \frac{u(x,0)}{s} = \frac{\sin(x)}{s}\\
&p^1: \quad \Phi_1(x,s) = \frac{s+\alpha(1-s)}{s}\left(\frac{\partial^2}{\partial x^2}\right)\Phi_0(x,s) = -\frac{s+\alpha(1-s)}{s}\sin(x)\\
&p^2: \quad \Phi_2(x,s) = \frac{s+\alpha(1-s)}{s}\left(\frac{\partial^2}{\partial x^2}\right)\Phi_1(x,s) = \frac{(s+\alpha(1-s))^2}{s^2}\sin(x)\\
&p^3: \quad \Phi_3(x,s) = \frac{s+\alpha(1-s)}{s}\left(\frac{\partial^2}{\partial x^2}\right)\Phi_2(x,s) = -\frac{(s+\alpha(1-s))^3}{s^4}\sin(x)\\
&\vdots\\
&p^i: \Phi_i(x,s) = \frac{(s+\alpha(1-s))}{s}\left(\frac{\partial^2}{\partial x^2}\right)\Phi_i(x,s) = (-1)^i\frac{(s+\alpha(1-s))^i}{s^{i+1}}\sin(x).
\end{aligned}$$

This implies that the approximate solution is then

$$U_n(x,s) = \sum_{i=0}^{n}\Phi_i(x,s) = \sin(x)\sum_{i=0}^{n}\frac{\left[-(s+\alpha(1-s))\right]^i}{s^{i+1}}, \tag{4.55}$$

giving

$$u_n(x,t) = \mathcal{L}^{-1}\left\{U_n(x,s)\right\} = \sin(x)\mathcal{L}^{-1}\left\{\sum_{i=0}^{n}\frac{\left[-(s+\alpha(1-s))\right]^i}{s^{i+1}}\right\}, \tag{4.56}$$

in the time-domain.

The approximate solutions for $n = 1, 5, 10$ for $\alpha = 0.1, 0.5, 0.9$ are shown in Fig. 4.8, 4.10, and 4.10 respectively. These results demonstrate the effectiveness of the LHAM method. As α reaches 0.9, we observe already a rapid convergence of the approximate solutions to the exact solution at an early stage in Fig. 4.10 contrarily to those shown in Fig. 4.7.

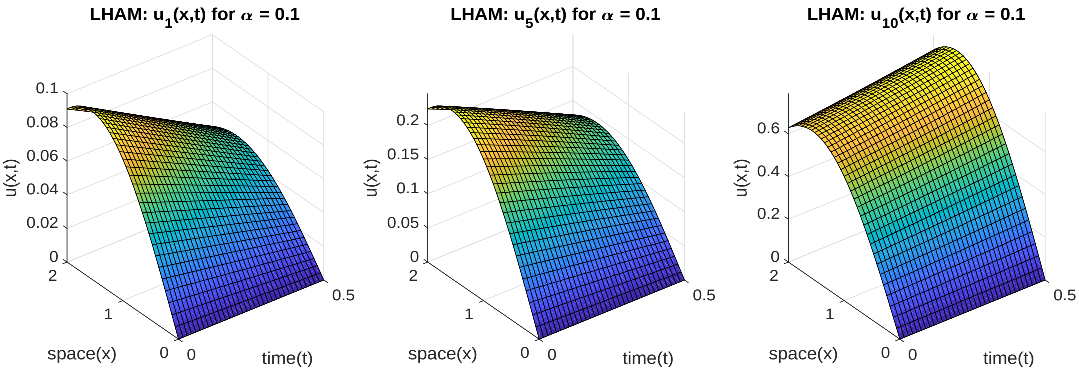

FIGURE 4.8: LHAM Approximate solution with fractional order ($\alpha = 0.1$) for $n = 1, 5, 10$.

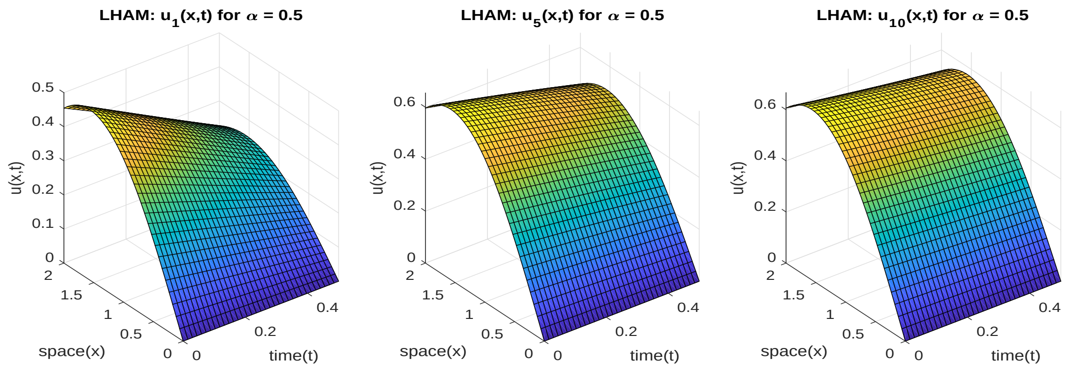

FIGURE 4.9: LHAM Approximate solution with fractional order ($\alpha = 0.5$) for $n = 1, 5, 10$.

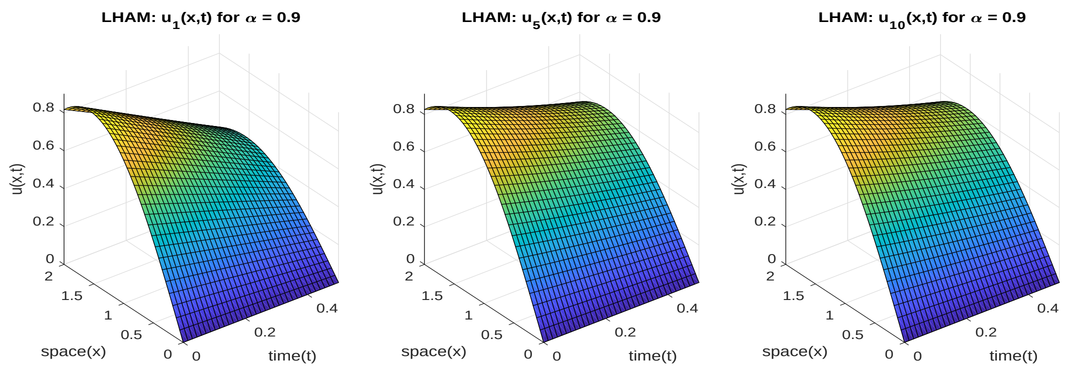

FIGURE 4.10: LHAM Approximate solution with fractional order ($\alpha = 0.9$) for $n = 1, 5, 10$.

In order to elucidate the convergence and stability related to LHAM, we express the exact analytical solution as $n \to \infty$ for the fractional order α in closed form as

$$u(x,t) = \sin(x) \sum_{i=0}^{\infty} \frac{(-t)^{i\alpha}}{\Gamma(i\alpha + 1)}, \tag{4.57}$$

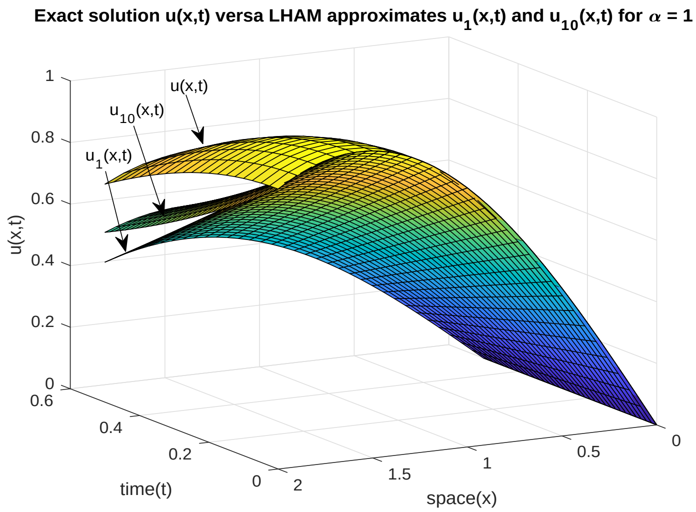

FIGURE 4.11: Exact solution $u(x,t)$ and approximate solutions: $u_1(x,t)$ and $u_{10}(x,t)$.

where $\sum_{i=0}^{\infty} \frac{(-t)^{i\alpha}}{\Gamma(i\alpha+1)}$ stands for $\lim_{n\to\infty} \mathcal{L}^{-1}\left\{\sum_{i=0}^{n} \frac{[-(s+\alpha(1-s))]^i}{s^{i+1}}\right\}$. Therefore the approximate analytical solution is obtained by

$$u_n(x,t) = \sin(x) \sum_{i=0}^{n} \frac{(-t)^{i\alpha}}{\Gamma(i\alpha+1)}. \tag{4.58}$$

Figure 4.11 shows the exact solution and two approximate solutions, $u_1(x,t)$ and $u_{10}(x,t)$. We observe a convergence effect as $u_{10}(x,t)$ is close to the exact solution as compared to $u_1(x,t)$. Hence, a minimum relative error is expected for approximate solution at large values of n. For $\alpha = 1$ in (4.57) , the exact analytical solution becomes

$$u(x,t) = e^{-t^2} sin(x). \tag{4.59}$$

In addition, Fig. 4.12 give a 2D- representation of how the solution changes with respect to different fractional order α. Here space ranges between 0 and 2 while the time is kept fixed, $t = 0.02$.

4.6.3 Shallow water waves application

Evolution equation could also be nonlinear depending on how complex is its operator A. Many models have been used to describe the behavior of the shallow water waves. This includes the generalized fifth order Korteweg-de Vries (KdV) equation defined by [47]

$$u_t + 45u^2 u_x + \lambda u_x u_{xx} + 15uu_{xxx} + u_{xxxxx} = 0, \tag{4.60}$$

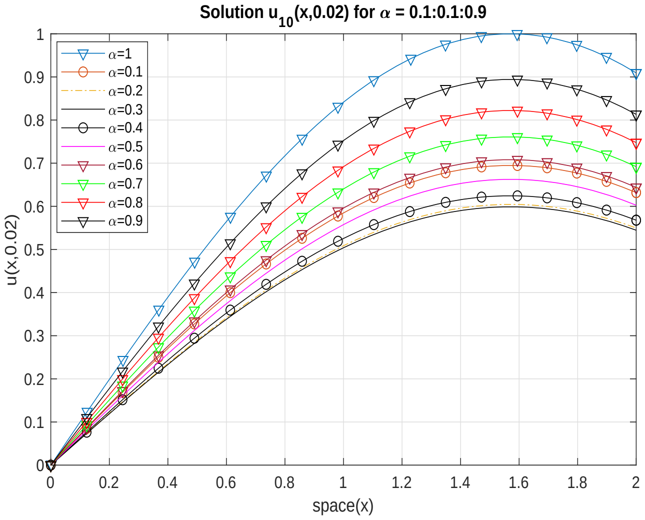

FIGURE 4.12: Approximate analytical solution at $n = 10$, for α values chosen arbitrary.

which for $\lambda = 15$, leads to one of the extensively studied equation called the Sawada-Kotera equation [48–50]. Its extension with CFFD reads as

$$\begin{cases} {}^{CF}D_t^{\alpha} u(t) + 45u^2 u_x + 15u_x u_{xx} + 15u u_{xxx} + u_{xxxxx} = 0 \\ u(x,0) = f(x), \end{cases} \tag{4.61}$$

where $Au(t) = -45u^2 u_x - 15u_x u_{xx} - 15u u_{xxx} - u_{xxxxx}$. The following analysis in terms of justifying the well-posedness of Equation (4.61) also presented in [42] is done according to the propositions made for generalized fractional evolution equation with non singular kernel. This goes as follows.

Theorem 4.21. *Consider the self-map $\mathcal{T}$ and redefine the model (4.61) as the iterative scheme*

$$\begin{aligned} \mathcal{T}(u_n(x,t)) &= u_{n+1}(x,t) \\ &= u_n(x,t) +^{CD} I_t^{\alpha}[-45u^2(x,\xi)u_x(x,\xi) - 15u_x(x,\xi)u_{xx}(x,\xi) \\ &\quad - 15u(x,\xi)u_{xxx}(x,\xi) - u_{xxxxx}(x,\xi)] \end{aligned}$$

then, it is $\mathcal{T}$-stable in $L^2(a,b)$.

Proof. First, we show the existence of a fixed point in $\mathcal{T}$. Let $i, j \in \mathbb{N}$ then,

$$\begin{aligned} \|\mathcal{T}u_i(x,t) - \mathcal{T}u_j(x,t)\| &= \|u_{i+1}(x,t) - u_{j+1}(x,t)\| \\ &= \Big\| u_i(x,t) +^{CF} I_t^{\alpha}[-45u_i^2 \partial_x u_i - 15\partial_x u_i \partial_{x^2}^2 u_i - 15u_i \partial_{x^3}^3 u_i + \partial_{x^5}^5 u_i] \\ &\quad -u_j(x,t) -^{CD} I_t^{\alpha}[-45u_j^2 \partial_x u_j - 15\partial_x u_j \partial_{x^2}^2 u_j - 15u_j \partial_{x^3}^3 u_j + \partial_{x^5}^5 u_j] \Big\| \end{aligned}$$

where

$$ {}^{CF}I_t^{\alpha}u(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t u(\xi)\,d\xi, $$

is the fractional integral associated to the Caputo-Fabrizio fractional derivative defined in (4.2). Then,

$$ \begin{aligned} \|\mathcal{T}u_i(x,t) - \mathcal{T}u_j(x,t)\| \leq{}& \|u_i(x,t)-u_j(x,t)\| - 45\|{}^{CF}I_t^{\alpha}[(u_i^2\partial_x u_i - u_j^2\partial_x u_j)]\| \\ & - 15\|{}^{CF}I_t^{\alpha}[(\partial^2_{x^2}u_i\partial_x u_i - \partial^2_{x^2}u_j\partial_x u_j)]\| - 15\|{}^{CF}I_t^{\alpha}[u_i\partial^3_{x^3}u_i - f_j\partial^3_{x^3}u_j]\| \\ & - \|{}^{CF}I_t^{\alpha}[\partial^5_{x^5}(u_i-u_j)]\| \\ \\ \leq{}& -\frac{90(1-\alpha)}{(2-\alpha)M(\alpha)}\delta_1 A_1\|u_i-u_j\| - \frac{90\alpha}{(2-\alpha)M(\alpha)}\delta_1 A_1 T\|u_i-u_j\| \\ & - \frac{30(1-\alpha)}{(2-\alpha)M(\alpha)}\delta_1\delta_2\delta_3\|u_i-u_j\| - \frac{30\alpha}{(2-\alpha)M(\alpha)}\delta_1\delta_2\delta_3 T\|u_i-u_j\| \\ & - \frac{30(1-\alpha)}{(2-\alpha)M(\alpha)}\vartheta_1\mathcal{K}_1\|u_i-u_j\| - \frac{30\alpha}{(2-\alpha)M(\alpha)}\vartheta_1\mathcal{K}_1 T\|u_i-u_j\| \\ & - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\kappa_1\kappa_2\kappa_3\kappa_4\kappa_5\|u_i-u_j\| - \frac{2\alpha}{(2-\alpha)M(\alpha)}\kappa_1\kappa_2\kappa_3\kappa_4\kappa_5 T\|u_i-u_j\|, \end{aligned} $$

or in simple form,

$$ \begin{aligned} \|\mathcal{T}u_i(x,t) - \mathcal{T}u_j(x,t)\| \leq{}& \frac{90}{(2-\alpha)M(\alpha)}\delta_1 A_1\left(\alpha(1+T)-1\right)\|u_i-u_j\| \\ & + \frac{30}{(2-\alpha)M(\alpha)}\delta_1\delta_2\delta_3\left(\alpha(1+T)-1\right)\|u_i-u_j\| \\ & + \frac{30}{(2-\alpha)M(\alpha)}\vartheta_1\mathcal{K}_1\left(\alpha(1+T)-1\right)\|u_i-u_j\| \\ & + \frac{2}{(2-\alpha)M(\alpha)}\kappa_1\kappa_2\kappa_3\kappa_4\kappa_5\left(\alpha(1+T)-1\right)\|u_i-u_j\|, \end{aligned} $$

where the Lipschitz condition for the differential operator ∂_x has been used together with the following positive constants $\kappa_i,\ i=1,2,3,4,5;\quad \delta_i,\ i=1,2,3\quad \vartheta_1$ and δ_1. Hence,

$$ \|\mathcal{T}u_i(x,t) - \mathcal{T}u_j(x,t)\| \leq L\|u_i(x,t)-u_j(x,t)\|, $$

where

$$ \begin{aligned} L ={}& \frac{90}{(2-\alpha)M(\alpha)}\delta_1 A_1\left(\alpha(1+T)-1\right) + \frac{30}{(2-\alpha)M(\alpha)}\delta_1\delta_2\delta_3\left(\alpha(1+T)-1\right) \\ & + \frac{30}{(2-\alpha)M(\alpha)}\vartheta_1\mathcal{K}_1\left(\alpha(1+T)-1\right) + \frac{2}{(2-\alpha)M(\alpha)}\kappa_1\kappa_2\kappa_3\kappa_4\kappa_5\left(\alpha(1+T)-1\right) \\ ={}& \frac{2}{(2-\alpha)M(\alpha)}\left(45\delta_1 A_1 + 15\delta_1\delta_2\delta_3 + 15\vartheta_1\mathcal{K}_1 + \kappa_1\kappa_2\kappa_3\kappa_4\kappa_5\right)\left(\alpha(1+T)-1\right), \end{aligned} $$

and this proves that the Lipschitz condition holds for the nonlinear operator $\mathcal{T}$ and hence, it has a fixed point.

Lastly, if we take $C = 0$ and $\gamma = L$ then, the conditions of Lemma 4.13 holds for $\mathcal{T}$ which is therefore picard $\mathcal{T}$-stable and the proof is complete. □

Remark 4.22. Theorem 4.21, comparable to Proposition 4.15 as applied to shallow water waves, shows the existence of a unique solution for the model (4.61) obtained via the fixed-point of $\mathcal{T}$ in the iterative scheme (4.21).

Furthermore, we present the numerical solution of CFFD based model (4.61). The application to shallow water wave as described in [42], is elaborated here. The operator of (4.61) is nonlinear. Therefore Laplace transform is not used but a recursive approach associated to the Sumudu transform.

Definition 4.23 (Sumudu Transform). The Sumudu transform is defined over the set of function $B = \left\{ f(t) | \, \exists \, M, \tau_1, \tau_2 > 0, \, |f(t)| < M e^{\frac{|t|}{\tau_i}}, \ if \ t \in (-1)^j \mathrm{x}[0, \infty) \right\}$, by

$$\mathcal{S}\left\{f(t), p\right\} = \int_0^\infty e^{-pt} f(pt) dt, \quad p \in (-\tau_1, \tau_2). \tag{4.62}$$

This transform can also be defined as follows [51]:

$$\mathcal{S}\left\{f(t), p\right\} = \frac{1}{p} \int_0^\infty e^{-\frac{t}{p}} f(pt) dt. \tag{4.63}$$

Further, Atangana [52], denounced the relationship existing between the Sumudu transform and CFFD given as

$$\mathcal{S}\left\{{}^{CF}D_t^\alpha u(x,t), p\right\} = M(\alpha) \frac{F(p) - f(0)}{1 - \alpha + \alpha p}, \tag{4.64}$$

where $F(p)$ stands for the Sumudu transform of $f(t)$.

For the purpose of solvability, we apply the Sumudu transform $\mathcal{S}$ on both sides of (4.61), which gives

$$\begin{aligned} \frac{U(x,p) - u(x,0)}{1 - \alpha + \alpha p} = \mathcal{S}\big\{ &- 45u^2(x,t)u_x(x,t) - 15u(x,t)u_{xxx}(x,t) - 15u_x(x,t)u_{xx}(x,t) \\ &- u_{xxxxx}(x,t) \big\}. \end{aligned}$$

When solving for $U(x,p)$, we have

$$\begin{aligned} U(x,p) = u(x,0) + (1 - \alpha + \alpha p)\mathcal{S}\big\{ &- 45u^2(x,t)u_x(x,t) - 15u(x,t)u_{xxx}(x,t) \\ &- 15u_x(x,t)u_{xx}(x,t) - u_{xxxxx}(x,t), p \big\}. \end{aligned} \tag{4.65}$$

Applying the inverse Sumudu transform $\mathcal{S}^{-1}$ to (4.65) leads to

$$u(x,t) = u(x,0) + \mathcal{S}^{-1}\big\{(1-\alpha+\alpha p)\mathcal{S}\big\{ -45u^2(x,t)u_x(x,t) - 15u(x,t)u_{xxx}(x,t) \\ - 15u_x(x,t)u_{xx}(x,t) - u_{xxxxx}(x,t), p\big\}\big\}. \tag{4.66}$$

From (4.66), an iterative formula is introduced as follows:

$$\begin{aligned} u(x,0) &= u_0 \\ u_{n+1}(x,t) &= u_0 + \mathcal{S}^{-1}\big\{(1-\alpha+\alpha p)\mathcal{S}\big\{ -45u_n^2\frac{\partial u_n}{\partial x} - 15u_n\frac{\partial^3 u_n}{\partial x^3} \\ &\qquad - 15\frac{\partial u_n}{\partial x}\frac{\partial^2 u_n}{\delta x^2} - \frac{\partial^5 u_n}{\partial x^5}, p\big\}\big\}. \end{aligned} \tag{4.67}$$

Consequently, the formula as defined in (4.67) gives an approximation for the solution of the Caputo-Fabrizio model (4.61). Hence

$$u(x,t) = \lim_{n\to\infty} u_n(x,t).$$

4.7 Summary

The main result of this dissertation is presented in this chapter. The well-posedness, which entails criteria such as existence, uniqueness and stability, was analyzed based on solution operators. It was shown that the generalized LEE with CFFD admitted a solution operator by proving the well-posedness of Problem (4.6) with aid of its Volterra version 4.8. This was done by proving the Lipschipz condition in Section 4.4 and the Picard $\mathcal{T}$-stability in Section 4.5. Three applications were presented in Section 4.6 to validate the fractional evolution equations with non-singular kernel. Laplace transform was applied in the first application to solve kinetic models, the Laplace homotopy analysis method in the second application to solve the linear diffusion model, and in the third application the Sumudu transform to solve the dispersion model which has a non-linear operator. Graphical illustrations were provided in Fig. 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.7, 4.8, 4.9, 4.10, 4.11 and 4.12. Matlab as a development software was selected for implementation.

Chapter 5

Conclusion and future work

"Mathematics is the science which draws necessary conclusions"

Benjamin Pierce

5.1 Conclusion

Throughout this dissertation, mathematical analysis of generalized linear evolution equations with the non-singular derivative was presented. Techniques used here showcased the well-posedness of the extended LEE with CFFD. Illustrative examples of applications validated this extension.

Firstly, we provided a literature review on fractional calculus. Basic fractional derivatives such as GLFD, RLFD and CFD were presented and their respective limitations discussed. Related properties governing these derivatives were also presented. The new CFFD was found appropriate since its derivative at order one was comparable to the classical derivative. For solvability purpose, Laplace transform of fractional derivatives were discussed. The concept of semigroups and solution operators were introduced.

Secondly, the concept of well-posedness as applied to generalized linear evolution equations has been elaborated in this dissertation. This included the demonstration of existence, uniqueness and stability of both ordinary and fractional types of evolution equations. Based on the fact that evolution models are described by a family of operators that form semigroups, then the theory of semigroup was first explored on different types of ODE for better understanding. We further analyzed the old Caputo-fractional evolution equation using the concept of solution operators which is just a generalization of semigroup theory. Under necessary and sufficient conditions proceeding from various definitions and theorems, the well-posedness of fractional evolution equation with CFD was established.

Thirdly, the generalized linear evolution equation was defined with the CFFD and its well-posedness investigated. It was demonstrated that the generalized LEE admits a solution operator by proving the existence, uniqueness and stability of a solution. This was done through both the Lipschitz condition and the Picard $\mathcal{T}$-stability. Further, we validated the extension of the CFFD to LEE in applications such as kinetic motion, heat diffusion, and dispersion of shallow water waves. Laplace transform was used in solving kinetic models and the LHAM for the linear diffusion model. It was also shown that though our analysis on LEE well-posedness could be applicable to nonlinear EE, the Sumudu transform was suitable to solve the nonlinear model for the dispersion of shallow water waves. Analysis results in each applications were provided.

Lastly, the mathematical analysis conducted in this dissertation has demonstrated that linear evolution equations are certainly extendable with the new Caput-Fabrizio fractional derivative. The results obtained in Chapter 4 justified that a generalized linear evolution equation has a solution, which is unique and stable with respect to associated parameters.

5.2 Future work

Although it was shown that an application of a nonlinear evolution equation also complied with the analysis approach presented for LEE, one could explore the concept of well-posedness for generalized nonlinear evolution equations. Since the proposed analysis only assumed for linear operators, then an investigation of the well-posedness for non-homogeneous fractional evolution equations with CFFD can be subject to future work.

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Appendix A

Matlab Codes

A.1 Matlab implementation of the Kinetic application

A.1.1 Case 1: M-file for Fig. 4.1

```
%%  Matlab Code for the stational kinetic model (K=0)
%% alpha = 0
t = 0:0.001:5;
a = 0;
Ma = 2./(2-a);
K = 0;
u0=1;
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b')
hold on
title('Solution of _{0}^{CF}D_{t}^\alpha u(t)=1  and u(0)=0 for \alpha=0:0.1:1')
%% alpha = 0.1
t = 0:0.001:5;
a = 0.1;
Ma = 2./(2-a);
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b','Linewidth',.3)
hold on
%% alpha = 0.2
t = 0:0.001:5;
a = 0.2;
Ma = 2./(2-a);
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b','Linewidth',.6)
hold on
```

```
%% alpha = 0.3
t = 0:0.001:5;
a = 0.3;
Ma = 2./(2-a);
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b','Linewidth',.9)
hold on
%% alpha = 0.4
t = 0:0.001:5;
a = 0.4;
Ma = 2./(2-a);
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b','Linewidth',1.2)
hold on
%% alpha =0.5
t = 0:0.001:5;
a = 0.5;
Ma = 2./(2-a);
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b','Linewidth',1.5)
hold on
%% alpha = 0.6
t = 0:0.001:5;
a = 0.6;
Ma = 2./(2-a);
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b','Linewidth',1.8)
hold on
%% alpha = 0.7
t = 0:0.001:5;
a = 0.7;
Ma = 2./(2-a);
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b','Linewidth',2.1)
hold on
%% alpha = 0.8
t = 0:0.001:5;
a = 0.8;
Ma = 2./(2-a);
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b','Linewidth',2.4)
hold on
%% alpha = 0.9
```

```
t = 0:0.001:5;
a = 0.9;
Ma = 2./(2-a);
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b','Linewidth',2.7)
ylabel('u(t)')
xlabel('t')
grid on
%% alpha = 0.1
t = 0:0.001:5;
a = 1;
Ma = 2./(2-a);
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b','Linewidth',3)
hold on
%%  end
```

A.1.2 Case 2: M-file for Fig. 4.2

```
%%  Matlab Code for the stational kinetic model (K=1)
%% alpha = 0
t = 0:0.001:5;
a = 0;
Ma = 2./(2-a);
K = 1;
u0=0;
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b')
hold on
title('Solution of _{0}^{CF}D_{t}^\alpha u(t)=1  and u(0)=0 for \alpha=0:0.1:1')
%% alpha = 0.1
t = 0:0.001:5;
a = 0.1;
Ma = 2./(2-a);
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b','Linewidth',.3)
hold on
%% alpha = 0.2
t = 0:0.001:5;
a = 0.2;
Ma = 2./(2-a);
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b','Linewidth',.6)
```

```
hold on
%% alpha = 0.3
t = 0:0.001:5;
a = 0.3;
Ma = 2./(2-a);
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b','Linewidth',.9)
hold on
%% alpha = 0.4
t = 0:0.001:5;
a = 0.4;
Ma = 2./(2-a);
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b','Linewidth',1.2)
hold on
%% alpha =0.5
t = 0:0.001:5;
a = 0.5;
Ma = 2./(2-a);
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b','Linewidth',1.5)
hold on
%% alpha = 0.6
t = 0:0.001:5;
a = 0.6;
Ma = 2./(2-a);
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b','Linewidth',1.8)
hold on
%% alpha = 0.7
t = 0:0.001:5;
a = 0.7;
Ma = 2./(2-a);
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b','Linewidth',2.1)
hold on
%% alpha = 0.8
t = 0:0.001:5;
a = 0.8;
Ma = 2./(2-a);
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b','Linewidth',2.4)
hold on
```

```
%% alpha = 0.9
t = 0:0.001:5;
a = 0.9;
Ma = 2./(2-a);
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b','Linewidth',2.7)
ylabel('u(t)')
xlabel('t')
grid on
%% alpha = 0.1
t = 0:0.001:5;
a = 1;
Ma = 2./(2-a);
M1 = (2*K-2*a*K)./(2.*Ma-a.*Ma);
M2 = (2.*a.*K)./(2.*Ma-a.*Ma);
ut = @(t)u0+M1.*K+M2.*K.*t;
plot(t,ut(t),'b','Linewidth',3)
hold on
%%  end
```

A.1.3 Case 3: M-file for Fig. 4.3

```
%%  Matlab Code for the stational kinetic model (K=1)
%% alpha = 0
t = 0:0.001:5;
a = 0;
Ma = 2./(2-a);
K = 1;
u0=1;
M1 = (2*Ma-a.*Ma)./((2.*Ma-a.*Ma)+2*K.*(1-a));
M2 = (2.*a.*K)./((2.*Ma-a.*Ma)+2*K.*(1-a));
ut = @(t)u0.*M1.*exp(-M2.*t);
plot(t,ut(t),'b')
hold on
title('Solution of _{0}^{CF}D_{t}^\alpha u(t)=Ku(t)  and u(0)=1 for \alpha
    =0:0.1:1')
%% alpha = 0.1
a = 0.1;
Ma = 2./(2-a);
M1 = (2*Ma-a.*Ma)./((2.*Ma-a.*Ma)+2*K.*(1-a));
M2 = (2.*a.*K)./((2.*Ma-a.*Ma)+2*K.*(1-a));
ut = @(t)u0.*M1.*exp(-M2.*t);
plot(t,ut(t),'b','Linewidth',.3)
hold on
%% alpha = 0.2
a = 0.2;
Ma = 2./(2-a);
M1 = (2*Ma-a.*Ma)./((2.*Ma-a.*Ma)+2*K.*(1-a));
M2 = (2.*a.*K)./((2.*Ma-a.*Ma)+2*K.*(1-a));
ut = @(t)u0.*M1.*exp(-M2.*t);
plot(t,ut(t),'b','Linewidth',.6)
```

```
hold on
%% alpha = 0.3
a = 0.3;
Ma = 2./(2-a);
M1 = (2*Ma-a.*Ma)./((2.*Ma-a.*Ma)+2*K.*(1-a));
M2 = (2.*a.*K)./((2.*Ma-a.*Ma)+2*K.*(1-a));
ut = @(t)u0.*M1.*exp(-M2.*t);
plot(t,ut(t),'b','Linewidth',.9)
hold on
%% alpha = 0.4
a = 0.4;
Ma = 2./(2-a);
M1 = (2*Ma-a.*Ma)./((2.*Ma-a.*Ma)+2*K.*(1-a));
M2 = (2.*a.*K)./((2.*Ma-a.*Ma)+2*K.*(1-a));
ut = @(t)u0.*M1.*exp(-M2.*t);
plot(t,ut(t),'b','Linewidth',1.2)
hold on
%% alpha = 0.5
a = 0.5;
Ma = 2./(2-a);
M1 = (2*Ma-a.*Ma)./((2.*Ma-a.*Ma)+2*K.*(1-a));
M2 = (2.*a.*K)./((2.*Ma-a.*Ma)+2*K.*(1-a));
ut = @(t)u0.*M1.*exp(-M2.*t);
plot(t,ut(t),'b','Linewidth',1.5)
hold on
%% alpha = 0.6
a = 0.6;
Ma = 2./(2-a);
M1 = (2*Ma-a.*Ma)./((2.*Ma-a.*Ma)+2*K.*(1-a));
M2 = (2.*a.*K)./((2.*Ma-a.*Ma)+2*K.*(1-a));
ut = @(t)u0.*M1.*exp(-M2.*t);
plot(t,ut(t),'b','Linewidth',1.8)
hold on
%% alpha = 0.7
a = 0.7;
Ma = 2./(2-a);
M1 = (2*Ma-a.*Ma)./((2.*Ma-a.*Ma)+2*K.*(1-a));
M2 = (2.*a.*K)./((2.*Ma-a.*Ma)+2*K.*(1-a));
ut = @(t)u0.*M1.*exp(-M2.*t);
plot(t,ut(t),'b','Linewidth',2.1)
hold on
%% alpha = 0.8
a = 0.8;
Ma = 2./(2-a);
M1 = (2*Ma-a.*Ma)./((2.*Ma-a.*Ma)+2*K.*(1-a));
M2 = (2.*a.*K)./((2.*Ma-a.*Ma)+2*K.*(1-a));
ut = @(t)u0.*M1.*exp(-M2.*t);
plot(t,ut(t),'b','Linewidth',2.4)
hold on
%% alpha = 0.9
a = 0.9;
Ma = 2./(2-a);
M1 = (2*Ma-a.*Ma)./((2.*Ma-a.*Ma)+2*K.*(1-a));
M2 = (2.*a.*K)./((2.*Ma-a.*Ma)+2*K.*(1-a));
ut = @(t)u0.*M1.*exp(-M2.*t);
```

```
plot(t,ut(t),'b','Linewidth',2.7)
ylabel('u(t)')
xlabel('t')
grid on
%% alpha = 1
a = 1;
Ma = 2./(2-a);
M1 = (2*Ma-a.*Ma)./((2.*Ma-a.*Ma)+2*K.*(1-a));
M2 = (2.*a.*K)./((2.*Ma-a.*Ma)+2*K.*(1-a));
ut = @(t)u0.*M1.*exp(-M2.*t);
plot(t,ut(t),'b','Linewidth',3)
hold on
%%  end
```

A.2 Matlab implementation of the Heat diffusion application

A.2.1 Exact solution of the heat equation for $\alpha = 1$ (Fig. 4.4)

```
%% For exact solution for fractional heat equation with CFFD
t = 0:0.025:.5;
x = 0:.1:2;
[xx,tt] = meshgrid(x,t);
ut = exp(-tt.^2).*sin(xx);
surf(tt,xx,ut)
xlabel('time(t)')
ylabel('space(x)')
zlabel('u(x,t)')
title('Exact Solution for \alpha =1')
%%   end
```

A.2.2 Approximate solution of the heat equation for $\alpha = 0.1$ and $n = 1, 5, 10$ (Fig. 4.5)

```
%% Approximate solution of the heat equation for $\alpha=0.1$ and n=1, 5, 10
clear all,clc
syms x,
syms t,
a = 0.1;
xmin = 0;
xmax = 2;
tmin = 0;
tmax = .5;
figure
%% For n = 0
U0 = sin(x);
%% For n = 1;
D0 = diff(U0,2);
```

```
simplify(D0);
I0 = int(D0,t);
simplify(I0);
U1 = U0 + (1-a)*D0-a*I0;
simplify(U1);
subplot(1,3,1)
fsurf(U1,[tmin,tmax,xmin,xmax])
xlabel('time(t)')
ylabel('space(x)')
zlabel('u(x,t)')
title('u_1(x,t) for \alpha = 0.1')
%% For n = 2;
D1 = diff(U1,2);
I1 = int(D1,t);
U2 = U0 + (1-a)*D1-a*I1;
%% For n = 3;
D2 = diff(U2,2);
I2 = int(D2,t);
U3 = U0 + (1-a)*D2-a*I2;
%% For n = 4;
D3 = diff(U3,2);
I3 = int(D3,t);
U4 = U0 + (1-a)*D3-a*I3;
%% For n=5
D4 = diff(U4,2);
I4 = int(D4,t);
U5 = U0 + (1-a)*D4-a*I4;
subplot(1,3,2)
fsurf(U5,[tmin,tmax,xmin,xmax])
xlabel('time(t)')
ylabel('space(x)')
zlabel('u(x,t)')
title('u_5(x,t) for \alpha = 0.1')
%% For n = 6;
D5 = diff(U5,2);
I5 = int(D5,t);
U6 = U0 + (1-a)*D5-a*I5;
%% For n = 7;
D6 = diff(U6,2);
I6 = int(D6,t);
U7 = U0 + (1-a)*D6-a*I6;
%% For n = 8;
D7 = diff(U7,2);
I7 = int(D7,t);
U8 = U0 + (1-a)*D7-a*I7;
%% For n = 9;
D8 = diff(U8,2);
I8 = int(D8,t);
U9 = U0 + (1-a)*D8-a*I8;
%% For n = 10;
D9 = diff(U9,2);
I9 = int(D9,t);
U10 = U0 + (1-a)*D9-a*I9;
subplot(1,3,3)
fsurf(U10,[tmin,tmax,xmin,xmax])
```

```
xlabel('time(t)')
ylabel('space(x)')
zlabel('u(x,t)')
title('u_1_0(x,t) for \alpha = 0.1')
%%  end
```

A.2.3 Approximate solution of the heat equation for $\alpha = 0.5$ and $n = 1, 5, 10$ (Fig. 4.6)

```
%% Approximate solution of the heat equation for $\alpha=0.5$ and n=1, 5, 10
clear all,clc
syms x,
syms t,
a = 0.5;
xmin = 0;
xmax = 2;
tmin = 0;
tmax = .5;
figure
%% For n = 0
U0 = sin(x);
%% For n = 1;
D0 = diff(U0,2);
simplify(D0);
I0 = int(D0,t);
simplify(I0);
U1 = U0 + (1-a)*D0-a*I0;
simplify(U1);
subplot(1,3,1)
fsurf(U1,[tmin,tmax,xmin,xmax])
xlabel('time(t)')
ylabel('space(x)')
zlabel('u(x,t)')
title('u_1(x,t) for \alpha = 0.5')
%% For n = 2;
D1 = diff(U1,2);
I1 = int(D1,t);
U2 = U0 + (1-a)*D1-a*I1;
%% For n = 3;
D2 = diff(U2,2);
I2 = int(D2,t);
U3 = U0 + (1-a)*D2-a*I2;
%% For n = 4;
D3 = diff(U3,2);
I3 = int(D3,t);
U4 = U0 + (1-a)*D3-a*I3;
%% For n=5
D4 = diff(U4,2);
I4 = int(D4,t);
U5 = U0 + (1-a)*D4-a*I4;
subplot(1,3,2)
fsurf(U5,[tmin,tmax,xmin,xmax])
```

```
xlabel('time(t)')
ylabel('space(x)')
zlabel('u(x,t)')
title('u_5(x,t) for \alpha = 0.5')
%% For n = 6;
D5 = diff(U5,2);
I5 = int(D5,t);
U6 = U0 + (1-a)*D5-a*I5;
%% For n = 7;
D6 = diff(U6,2);
I6 = int(D6,t);
U7 = U0 + (1-a)*D6-a*I6;
%% For n = 8;
D7 = diff(U7,2);
I7 = int(D7,t);
U8 = U0 + (1-a)*D7-a*I7;
%% For n = 9;
D8 = diff(U8,2);
I8 = int(D8,t);
U9 = U0 + (1-a)*D8-a*I8;
%% For n = 10;
D9 = diff(U9,2);
I9 = int(D9,t);
U10 = U0 + (1-a)*D9-a*I9;
subplot(1,3,3)
fsurf(U10,[tmin,tmax,xmin,xmax])
xlabel('time(t)')
ylabel('space(x)')
zlabel('u(x,t)')
title('u_1_0(x,t) for \alpha = 0.5')
%%  end
```

A.2.4 Approximate solution of the heat equation for $\alpha = 0.9$ and $n = 1, 5, 10$ (Fig. 4.7)

```
%% Approximate solution of the heat equation for $\alpha=0.1$ and n=1, 5, 10
clear all,clc
syms x,
syms t,
a = 0.9;
xmin = 0;
xmax = 2;
tmin = 0;
tmax = .5;
figure
%% For n = 0
U0 = sin(x);
%% For n = 1;
D0 = diff(U0,2);
simplify(D0);
I0 = int(D0,t);
simplify(I0);
```

```
U1 = U0 + (1-a)*D0-a*I0;
simplify(U1);
subplot(1,3,1)
fsurf(U1,[tmin,tmax,xmin,xmax])
xlabel('time(t)')
ylabel('space(x)')
zlabel('u(x,t)')
title('u_1(x,t) for \alpha = 0.9')
%% For n = 2;
D1 = diff(U1,2);
I1 = int(D1,t);
U2 = U0 + (1-a)*D1-a*I1;
%% For n = 3;
D2 = diff(U2,2);
I2 = int(D2,t);
U3 = U0 + (1-a)*D2-a*I2;
%% For n = 4;
D3 = diff(U3,2);
I3 = int(D3,t);
U4 = U0 + (1-a)*D3-a*I3;
%% For n=5
D4 = diff(U4,2);
I4 = int(D4,t);
U5 = U0 + (1-a)*D4-a*I4;
subplot(1,3,2)
fsurf(U5,[tmin,tmax,xmin,xmax])
xlabel('time(t)')
ylabel('space(x)')
zlabel('u(x,t)')
title('u_5(x,t) for \alpha = 0.9')
%% For n = 6;
D5 = diff(U5,2);
I5 = int(D5,t);
U6 = U0 + (1-a)*D5-a*I5;
%% For n = 7;
D6 = diff(U6,2);
I6 = int(D6,t);
U7 = U0 + (1-a)*D6-a*I6;
%% For n = 8;
D7 = diff(U7,2);
I7 = int(D7,t);
U8 = U0 + (1-a)*D7-a*I7;
%% For n = 9;
D8 = diff(U8,2);
I8 = int(D8,t);
U9 = U0 + (1-a)*D8-a*I8;
%% For n = 10;
D9 = diff(U9,2);
I9 = int(D9,t);
U10 = U0 + (1-a)*D9-a*I9;
subplot(1,3,3)
fsurf(U10,[tmin,tmax,xmin,xmax])
xlabel('time(t)')
ylabel('space(x)')
zlabel('u(x,t)')
```

```
title('u_1_0(x,t) for \alpha = 0.9')
%%  end
```

A.2.5 LHAM Approximate solution of the heat equation for $\alpha = 0.1$ and $n = 1, 5, 10$ (Fig. 4.8)

```
%% LHAM Approximate solution of the heat equation for $\alpha=0.1$ and n=1, 5, 10
clear all,clc
syms x,
syms t,
syms s
syms m
a = 0.1;
xmin = 0;
xmax = 2;
tmin = 0;
tmax = .5;
%% For n = 1;
n = 1;
Un = sin(x)*symsum(((-1)^m)*((s+a*(1-s))^m)/(s^(m+1)),m,0,n);
u_n = ilaplace(Un);
subplot(1,3,1)
fsurf(u_n,[tmin,tmax,xmin,xmax])
xlabel('time(t)')
ylabel('space(x)')
zlabel('u(x,t)')
title('LHAM: u_1(x,t) for \alpha = 0.1')
%% For n=5
n = 5;
Un = sin(x)*symsum(((-1)^m)*((s+a*(1-s))^m)/(s^(m+1)),m,0,n);
u_n = ilaplace(Un);
subplot(1,3,2)
fsurf(u_n,[tmin,tmax,xmin,xmax])
xlabel('time(t)')
ylabel('space(x)')
zlabel('u(x,t)')
title('u_5(x,t)')
title('LHAM: u_5(x,t) for \alpha = 0.1')
%% For n = 10;
n = 10;
Un = sin(x)*symsum(((-1)^m)*((s+a*(1-s))^m)/(s^(m+1)),m,0,n);
u_n = ilaplace(Un);
subplot(1,3,3)
fsurf(u_n,[tmin,tmax,xmin,xmax])
xlabel('time(t)')
ylabel('space(x)')
zlabel('u(x,t)')
title('LHAM: u_1_0(x,t) for \alpha = 0.1')
%%  end
```

A.2.6 LHAM Approximate solution of the heat equation for $\alpha = 0.5$ and $n = 1, 5, 10$ (Fig. 4.9)

```
%% LHAM Approximate solution of the heat equation for $\alpha=0.5$ and n=1, 5, 10
clear all,clc
syms x,
syms t,
syms s
syms m
a = 0.5;
xmin = 0;
xmax = 2;
tmin = 0;
tmax = .5;
%% For n = 1;
n = 1;
Un = sin(x)*symsum(((-1)^m)*((s+a*(1-s))^m)/(s^(m+1)),m,0,n);
u_n = ilaplace(Un);
subplot(1,3,1)
fsurf(u_n,[tmin,tmax,xmin,xmax])
xlabel('time(t)')
ylabel('space(x)')
zlabel('u(x,t)')
title('LHAM: u_1(x,t) for \alpha = 0.5')
%% For n=5
n = 5;
Un = sin(x)*symsum(((-1)^m)*((s+a*(1-s))^m)/(s^(m+1)),m,0,n);
u_n = ilaplace(Un);
subplot(1,3,2)
fsurf(u_n,[tmin,tmax,xmin,xmax])
xlabel('time(t)')
ylabel('space(x)')
zlabel('u(x,t)')
title('u_5(x,t)')
title('LHAM: u_5(x,t) for \alpha = 0.5')
%% For n = 10;
n = 10;
Un = sin(x)*symsum(((-1)^m)*((s+a*(1-s))^m)/(s^(m+1)),m,0,n);
u_n = ilaplace(Un);
subplot(1,3,3)
fsurf(u_n,[tmin,tmax,xmin,xmax])
xlabel('time(t)')
ylabel('space(x)')
zlabel('u(x,t)')
title('LHAM: u_1_0(x,t) for \alpha = 0.5')
%%  end
```

A.2.7 LHAM Approximate solution of the heat equation for $\alpha = 0.9$ and $n = 1, 5, 10$ (Fig. 4.10)

```
%% LHAM Approximate solution of the heat equation for $\alpha=0.5$ and n=1, 5, 10
clear all,clc
```

```
syms x,
syms t,
syms s
syms m
a = 0.9;
xmin = 0;
xmax = 2;
tmin = 0;
tmax = .5;
%% For n = 1;
n = 1;
Un = sin(x)*symsum(((-1)^m)*((s+a*(1-s))^m)/(s^(m+1)),m,0,n);
u_n = ilaplace(Un);
subplot(1,3,1)
fsurf(u_n,[tmin,tmax,xmin,xmax])
xlabel('time(t)')
ylabel('space(x)')
zlabel('u(x,t)')
title('LHAM: u_1(x,t) for \alpha = 0.9')
%% For n=5
n = 5;
Un = sin(x)*symsum(((-1)^m)*((s+a*(1-s))^m)/(s^(m+1)),m,0,n);
u_n = ilaplace(Un);
subplot(1,3,2)
fsurf(u_n,[tmin,tmax,xmin,xmax])
xlabel('time(t)')
ylabel('space(x)')
zlabel('u(x,t)')
title('u_5(x,t)')
title('LHAM: u_5(x,t) for \alpha = 0.9')
%% For n = 10;
n = 10;
Un = sin(x)*symsum(((-1)^m)*((s+a*(1-s))^m)/(s^(m+1)),m,0,n);
u_n = ilaplace(Un);
subplot(1,3,3)
fsurf(u_n,[tmin,tmax,xmin,xmax])
xlabel('time(t)')
ylabel('space(x)')
zlabel('u(x,t)')
title('LHAM: u_1_0(x,t) for \alpha = 0.9')
%%  end
```

A.2.8 Comparing the exact solution and the LHAM Approximate solution of the heat equation for $\alpha = 1$ and $n = 1, 10$ (Fig. 4.11)

```
%%  Exact solution against LHAM approximate solution for n =1, 10
%%  Exact analytical solution for alpha = 1
clear all,clc
t = 0:0.025:.5;
x = 0:.1:2;
[xx,tt] = meshgrid(x,t);
ut = exp(-tt.^2).*sin(xx);
```

```
surf(tt,xx,ut)
hold on
%% LHAM Approximate solution for alpha = 1 and n = 1,
clear all,clc
syms x,
syms t,
syms s
syms m
a = 1;
xmin = 0;
xmax = 2;
tmin = 0;
tmax = .5;
n = 1;
u_n = sin(x)*symsum((-t)^(m*a)/(gamma(m*a+1)),m,0,n)
hold on
fsurf(u_n,[tmin,tmax,xmin,xmax])
%% LHAM Approximate solution for alpha = 1 and n = 10
n = 10;
u_n = sin(x)*symsum((-t)^(m*a)/(gamma(m*a+1)),m,0,n)
hold on
fsurf(u_n,[tmin,tmax,xmin,xmax])
xlabel('time(t)')
ylabel('space(x)')
zlabel('u(x,t)')
title('Exact solution u(x,t) versa LHAM approximates u_1(x,t) and u_1_0(x,t) for
    \alpha = 1')
%% end
```

A.2.9 LHAM Approximate solution of the heat equation for $\alpha = 0.1 : 0.1 : 0.9$ and $n = 1$ (Fig. 4.12)

```
%% LHAM Approximate solution for n = 10 and alpha = 0.1:0.1:0.9
clear all,clc
syms x,
syms t,
syms s
syms m
xmin = 0;
xmax = 2;
tmin = 0;
tmax = .5;
tvar = .02;
% a = 1
a = 1;
n = 10;
Un = sin(x)*symsum(((-1)^m)*((s+a*(1-s))^m)/(s^(m+1)),m,0,n)
u_n = ilaplace(Un)
u_n = subs(u_n,t,0)
fplot(u_n,[xmin,xmax],'v-')
%% For alpha = 0.1
a = 0.1;
```

```
n = 10;
Un = sin(x)*symsum(((-1)^m)*((s+a*(1-s))^m)/(s^(m+1)),m,0,n)
u_n = ilaplace(Un)
u_n = subs(u_n,t,tvar)
hold on
fplot(u_n,[xmin,xmax],'-o')
%% For alpha = 0.2
a = 0.2;
n = 10;
Un = sin(x)*symsum(((-1)^m)*((s+a*(1-s))^m)/(s^(m+1)),m,0,n)
u_n = ilaplace(Un)
u_n = subs(u_n,t,tvar)
fplot(u_n,[xmin,xmax],'-.')
%% For alpha = 0.3
a = 0.3;
n = 10;
Un = sin(x)*symsum(((-1)^m)*((s+a*(1-s))^m)/(s^(m+1)),m,0,n)
u_n = ilaplace(Un)
u_n = subs(u_n,t,tvar)
fplot(u_n,[xmin,xmax],'k')
%% For alpha = 0.4
a = 0.4;
n = 10;
Un = sin(x)*symsum(((-1)^m)*((s+a*(1-s))^m)/(s^(m+1)),m,0,n)
u_n = ilaplace(Un)
u_n = subs(u_n,t,tvar)
fplot(u_n,[xmin,xmax],'-ko')
%% For alpha = 0.5
a = 0.5;
n = 10;
Un = sin(x)*symsum(((-1)^m)*((s+a*(1-s))^m)/(s^(m+1)),m,0,n)
u_n = ilaplace(Un)
u_n = subs(u_n,t,tvar)
fplot(u_n,[xmin,xmax],'-m')
%% For alpha = 0.6
a = 0.6;
n = 10;
Un = sin(x)*symsum(((-1)^m)*((s+a*(1-s))^m)/(s^(m+1)),m,0,n)
u_n = ilaplace(Un)
u_n = subs(u_n,t,tvar)
fplot(u_n,[xmin,xmax],'-v')
%% For alpha = 0.7
a = 0.7;
n = 10;
Un = sin(x)*symsum(((-1)^m)*((s+a*(1-s))^m)/(s^(m+1)),m,0,n)
u_n = ilaplace(Un)
u_n = subs(u_n,t,tvar)
fplot(u_n,[xmin,xmax],'-gv')
%% For alpha = 0.8
a = 0.8;
n = 10;
Un = sin(x)*symsum(((-1)^m)*((s+a*(1-s))^m)/(s^(m+1)),m,0,n)
u_n = ilaplace(Un)
u_n = subs(u_n,t,tvar)
fplot(u_n,[xmin,xmax],'-rv')
```

```
%% For alpha = 0.9
a = 0.9;
n = 10;
Un = sin(x)*symsum(((-1)^m)*((s+a*(1-s))^m)/(s^(m+1)),m,0,n)
u_n = ilaplace(Un)
u_n = subs(u_n,t,tvar)
fplot(u_n,[xmin,xmax],'-kv')
xlabel('space(x)')
ylabel('u(x,0.02)')
title('Solution u_1_0(x,0.02) for \alpha = 0.1:0.1:0.9')
grid on
%%  end
```